Hyperbolic Geometry and Schwarz–Pick Type Estimates for Harmonic Quasiconformal Mappings

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Abstract. We discuss several Schwarz–Pick type estimates for harmonic quasiconformal mappings through the geometry of conformal metrics of negative curvature. The starting point is the classical idea that holomorphic mappings between hyperbolic surfaces decrease the hyperbolic distance; our aim is to understand to what extent analogous phenomena remain valid in the harmonic and quasiconformal setting. A central role is played by sharp estimates for derivatives of the density of the hyperbolic metric on simply connected plane domains, in particular by the estimate

$$|(\log \lambda_G)_{zz}(z)| \leq \frac{7}{2}\lambda_G(z),$$

which is sharp and is attained for the slit plane $G = \mathbb{C}\backslash[0, +\infty)$. These estimates allow one to control the Gaussian curvature of the pull-back type metric

$$\lambda_G(f(z))|f_z(z)|^2\,|dz|^2$$

associated with a harmonic quasiconformal mapping $f : \mathbb{D} \to G$. As a consequence, one obtains hyperbolic Lipschitz and co-Lipschitz estimates, including improved constants under the natural completeness assumption for the corresponding conformal metric. We also explain how these estimates lead to Koebe type distortion results for harmonic quasiconformal mappings with simply connected ranges. In the final part of the talk, we consider real-valued harmonic mappings into $(-1, 1)$, as well as related estimates on Riemann surfaces equipped with complete conformal metrics whose Gaussian curvature is bounded from below by a negative constant. The results fit into the broader circle of ideas surrounding the Ahlfors–Schwarz lemma, Royden's Schwarz lemma and Yau's general Schwarz lemma, and they continue the line of work of Mateljević, Knežević and collaborators on hyperbolic metrics, harmonic quasiconformal mappings and geometric distortion estimates.

Keywords. Hyperbolic metric; Schwarz–Pick lemma; harmonic quasiconformal mappings; Gaussian curvature; conformal metrics; Koebe theorem; Riemann surfaces.

Selected references. Ahlfors, *Conformal Invariants*; Royden, Schwarz lemma type results for Kähler manifolds; Yau, *A general Schwarz lemma for Kähler manifolds*; Mateljević, works on the Schwarz lemma, curvature and distance; Knežević–Mateljević, quasi-isometries of harmonic quasiconformal mappings; Knežević–Mateljević–Svetlik, recent estimates for hyperbolic derivatives of harmonic quasiconformal mappings.