

UNIVERSITY OF BELGRADE
FACULTY OF MATHEMATICS

Katarina Bogdanović

JOINT SPECTRAL RADIUS OF THE
SCHUR-HADAMARD PRODUCT OF SET OF
MATRICES AND SCHUR-HADAMARD
MULTIPLIERS WITH APPLICATION TO
DERIVATION NORM INEQUALITIES FOR
OPERATORS

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УНИВЕРЗИТЕТ У БЕОГРАДУ
МАТЕМАТИЧКИ ФАКУЛТЕТ

Катарина Богдановић

ЗАЈЕДНИЧКИ СПЕКТРАЛНИ РАДИЈУС
ШУР-АДАМАРОВОГ ПРОИЗВОДА СКУПА
МАТРИЦА И ШУР-АДАМАРОВИ МНОЖИОЦИ
СА ПРИМЕНАМА НА ДЕРИВАЦИОНЕ НОРМА
НЕЈЕДНАКОСТИ ЗА ОПЕРАТОРЕ

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Ментор:

др Данко Р. ЈОЦИЋ, редовни професор
Универзитет у Београду, Математички факултет,
dr Aljoša ПЕРЕРКО, associate professor
University of Ljubljana, Faculty of Mechanical Engineering

Чланови комисије:

dr Fabian WIRTH, full professor
University of Passau, Faculty of Computer Science and Mathematics,
Germany

dr Roman DRNOVŠEK, full professor
University of Ljubljana, Faculty of Mathematics and Physics

др Драган ЂОРЂЕВИЋ, редовни професор
Универзитет у Нишу, Природно-математички факултет

др Стефан У. МИЛОШЕВИЋ, доцент
Универзитет у Београду, Математички факултет

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Dissertation title: Joint spectral radius of the Shur-Hadamard product of set of matrices and Schur-Hadamard multipliers with application to derivation norm inequalities for operators

Abstract: In the first and the second chapter of dissertation we prove some new inequalities for the spectral radius, essential spectral radius, operator norm, measure of non-compactness and numerical radius of Hadamard (Schur) weighted geometric means of positive kernel operators on Banach function and sequence spaces. The list of extensions and refinings of known inequalities has been expanded. Some new inequalities and equalities for the generalized and the joint spectral radius and their essential versions of Hadamard (Schur) geometric means of bounded sets of positive kernel operators on Banach function spaces have been proved. There are additional results in case of non-negative matrices that define operators on Banach sequence spaces. In the third part we present some inequalities for operator monotone functions and (co)hyponormal operators and give relations of Schur multipliers to derivation like inequalities for operators. In particular, let Ψ, Φ be s.n. functions, $p \geq 2$ and φ be an operator monotone function on $[0, \infty)$ such that $\varphi(0) = 0$. If $A, B, X \in \mathcal{B}(\mathcal{H})$ and A and B are strictly accretive such that $AX - XB \in \mathfrak{C}_\Psi(\mathcal{H})$, then also $AX\varphi(B) - \varphi(A)XB \in \mathfrak{C}_\Psi(\mathcal{H})$ and

$$\|AX\varphi(B) - \varphi(A)XB\|_\Psi \leq \left\| \sqrt{\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)\left(\frac{A+A^*}{2}\right)^{-1}} \right. \\ \left. A(AX - XB)B\left(\frac{B+B^*}{2}\right)^{-1} \sqrt{\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)} \right\|_\Psi.$$

under any of the following conditions:

- (a) Both A and B are normal,
- (b) A is cohyponormal, B is hyponormal and at least one of them is normal, and $\Psi := \Phi^{(p)*}$,
- (c) A is cohyponormal, B is hyponormal and $\|\cdot\|_\Psi$ is the trace norm $\|\cdot\|_1$.

Alternative inequalities for $\|\cdot\|_{\Psi^{(p)}}$ norms are also obtained.

Keywords: operator monotone functions, hyponormal operators, compact

operators, Schur-Hadamard weighted geometric mean ,kernel operators ,
joint spectral radius

Research area: mathematics

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Наслов дисертације: Заједнички спектрални радијус Шур-Адамаровог производа скупа матрица и Шур-Адамарови множиоци са применама на деривационе норма неједнакости за операторе

Резиме: У првом и другом поглављу дисертације доказујемо нове неједнакости за спектрални радијус, есенцијални спектрални радијус, норму оператора, меру некомпактности и нумерички радијус Адамарове (Шурове) тежинске геометријске средине позитивних интегралних оператора на Банаховим функцијском просторима и просторима низова. Проширена је листа екстензија и рафинација познатих неједнакости. Доказане су неке нове неједнакости и једнакости за генерализовани и заједнички спектрални радијус Адамарове (Шурове) геометријске средине ограничених скупова позитивних интегралних оператора на Банаховим функцијским просторима и њихове есенцијалне верзије. Има и додатних резултата у случају ненегативних матрица које дефинишу операторе на Банаховим просторима низова. У трећем делу презентоване су неке неједнакости за оператор монотоне функције и (ко)хипонормалне операторе и дат је однос Шупових множиоца у неједнакостима за операторе деривационог типа. Посебно, нека су Ψ, Φ с.н. функције, $p \geq 2$ и φ оператор монотона функција на $[0, \infty)$ таква да $\varphi(0) = 0$. Ако су $A, B, X \in \mathfrak{B}(\mathcal{H})$ и A и B строго акретивни такви да $AX - XB \in \mathfrak{C}_\Psi(\mathcal{H})$, онда следи да $AX\varphi(B) - \varphi(A)XB \in \mathfrak{C}_\Psi(\mathcal{H})$ и

$$\|AX\varphi(B) - \varphi(A)XB\|_\Psi \leq \left\| \sqrt{\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)} \left(\frac{A+A^*}{2}\right)^{-1} \right. \\ \left. A(AX - XB)B\left(\frac{B+B^*}{2}\right)^{-1} \sqrt{\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)} \right\|_\Psi.$$

Ако су испуњени следећи услови:

- (а) Оба A и B су нормални,
- (б) A је кохипонормалан, B је хипонормалан и барем један од њих је нормалан, и $\Psi := \Phi^{(p)*}$,
- (в) A је кохипонормалан, B је хипонормалан и $\|\cdot\|_\Psi$ је нуклеарна норма $\|\cdot\|_1$.

Добијене су и алтернативне неједнакости за $\|\cdot\|_{\Psi^{(p)}}$ норме.

Кључне речи: оператор монотоне функције, хипонормални оператори, компактни оператори, Шур-Адамарова тежинска геометријска средина, интегрални оператори, заједнички спектрални радијус

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

It is well known that entrywise Hadamard-Schur product $\circ$ of matrices (or more general kernel operators) plays an important role in operator theory and matrix analysis, for instance in the theories of operator means, Schur multipliers, preservers (e.g in tropical mathematics) and elsewhere, see e.g. [17, 44]. As pointed out in [17] the application of Hadamard-Schur product is a common and powerful technique in investigation of general matrix (and/or operator) norm inequalities, and particularly so in that of perturbation inequalities and commutator estimates. Assumed that $n \times n$ complex matrices H , K and X are given with H and K positive (definite) and write diagonalizations $H = U \text{diag}(s_1, s_2, \dots, s_n) U^*$ and $K = V \text{diag}(t_1, t_2, \dots, t_n) V^*$. To a given scalar mean $M(s, t)$ (for $s, t \geq 0$) one can associate the corresponding matrix mean $M(H, K)X$ by

$$M(H, K)X = U([M(s_i, t_j)] \circ (U^* X V)) V^*.$$

For a scalar mean $M(s, t)$ of the form $\sum_{i=1}^n f_i(s)g_i(t)$ one can observe that $M(H, K)X = \sum_{i=1}^n f_i(H)Xg_i(K)$ and we note that this expression makes a perfect sense even for Hilbert space operators H, K, X with H, K positive (semidefinite). However, for the definition of more general matrix means (such as interpolation means and binomial means) the use of Hadamard products or something alike seems unavoidable. In the development of operator means theory [17] several classical tools are used such as kernel oper-

ator representation of Hilbert-Schmidt operators and Schur multipliers, but also more involved tools such as (Stieltjes) double integral transformations and Grothendieck theorem (see [17]).

Let μ be a σ -finite positive measure on a σ -algebra $\mathcal{M}$ of subsets of a non-void set X . Let $M(X, \mu)$ be the vector space of all equivalence classes of (almost everywhere equal) complex measurable functions on X . A Banach space $L \subseteq M(X, \mu)$ is called a Banach function space if $f \in L$, $g \in M(X, \mu)$, and $|g| \leq |f|$ imply that $g \in L$ and $\|g\| \leq \|f\|$. It is assumed that X is the carrier of L , that is, there is no subset Y of X of strictly positive measure with the property that $f = 0$ a.e. on Y for all $f \in L$. By R we denote the set $\mathbb{N}$ of all natural numbers or the set $\{1, \dots, N\}$ for some $N \in \mathbb{N}$. Let $S(R)$ be the vector lattice of all complex sequences $(x_n)_{n \in R}$. A Banach space $L \subseteq S(R)$ is called a Banach sequence space if $x \in S(R)$, $y \in L$ and $|x| \leq |y|$ imply that $x \in L$ and $\|x\|_L \leq \|y\|_L$. Observe that a Banach sequence space is a Banach function space over a measure space (R, μ) , where μ denotes the counting measure on R . Denote by $\mathcal{L}$ the collection of all Banach sequence spaces L satisfying the property that $e_n = \chi_{\{n\}} \in L$ and $\|e_n\|_L = 1$ for all $n \in R$. For $L \in \mathcal{L}$ the set R is the carrier of L .

The cartesian product $L = E \times F$ of Banach function spaces is again a Banach function space, with the norm $\|(f, g)\|_L = \max\{\|f\|_E, \|g\|_F\}$.

Standard examples of Banach sequence spaces are Euclidean spaces, l^p spaces for $1 \leq p \leq \infty$, the space $c_0 \in \mathcal{L}$ of all null convergent sequences (equipped with the usual norms and the counting measure), while standard examples of Banach function spaces are the well-known spaces $L^p(X, \mu)$ ($1 \leq p \leq \infty$) and other less known examples such as Orlicz, Lorentz, Marcinkiewicz and more general rearrangement-invariant spaces, which are important e.g. in interpolation theory and in the theory of partial differential equations ([1]).

The cone of positive elements in L is denoted by L_+ . A non-negative function $f \in L_+$ is said to be strictly positive if $f(x) > 0$ for almost all $x \in X$.

By $M(X, \mu)_+$ we denote the cone of all equivalence classes of (almost everywhere equal) μ -measurable functions on X whose values lie in $[0, \infty]$. A subset $A \subset M(X, \mu)_+$ is said to be solid, if $f \leq g$ a.e., $f \in M(X, \mu)_+$, $g \in A$ implies $f \in A$.

A mapping $h : M(X, \mu)_+ \rightarrow [0, \infty]$ is a *function seminorm* if the following conditions hold for all $f, g \in M(X, \mu)_+$ and $\alpha \geq 0$:

- 1.) If $f \leq g$ a.e., then $h(f) \leq h(g)$,
- 2.) $h(f + g) \leq h(f) + h(g)$,
- 3.) $h(\alpha f) = \alpha h(f)$.

For a function seminorm h we define $C_h = \{f \in M(X, \mu)_+ : h(f) < \infty\}$, which is a solid subcone of $M(X, \mu)_+$. If, in addition, $h(f) = 0$ implies $f = 0$ a.e., then h is said to be a *function norm*.

A vector subspace $L \subseteq M(X, \mu)$ is called an *ideal* if $f \in M(X, \mu)$, $g \in L$ and $|f| \leq |g|$ a.e. imply that $f \in L$. It is assumed that X is the carrier of a such ideal $L \subseteq M(X, \mu)$.

A seminorm ρ on the ideal $L \subseteq M(X, \mu)$ is called a *lattice seminorm* (also *Riesz seminorm*) if $f \in M(X, \mu)$, $g \in L$ and $|f| \leq |g|$ a.e. imply that $\rho(f) \leq \rho(g)$. A *lattice norm* is a lattice seminorm which is also a norm. An ideal $L \subseteq M(X, \mu)$ equipped with a lattice norm ρ is sometimes called a *normed Köthe space* ([45, p. 421]) and that a complete normed Köthe space coincides with the Banach function space defined above.

If $\{f_n\}_{n \in \mathbb{N}} \subset M(X, \mu)$ is a decreasing real sequence and $f = \inf\{f_n \in M(X, \mu) : n \in \mathbb{N}\}$, then we write $f_n \downarrow f$. A Banach function space L has an *order continuous norm*, if $0 \leq f_n \downarrow 0$ implies $\|f_n\|_L \rightarrow 0$ as $n \rightarrow \infty$. The spaces $L^p(X, \mu)$, $1 \leq p < \infty$ have order continuous norm. Moreover, every reflexive Banach function space has an order continuous norm. In particular, we will be interested in Banach function spaces L such that L and its Banach dual space L^* have order continuous norms. Examples of such spaces are $L^p(X, \mu)$, $1 < p < \infty$, while the space $L = c_0$ is an example of a non-reflexive Banach sequence space, such that L and $L^* = l^1$ have order continuous norms.

By an operator on a Banach function space L we always mean a linear operator on L . An operator A on L is said to be *positive* if it maps non-negative functions to nonnegative ones, i.e., $AL_+ \subset L_+$, where L_+ denotes the positive cone $L_+ = \{f \in L : f \geq 0 \text{ a.e.}\}$. Given operators A and B on L , we write $A \geq B$ if the operator $A - B$ is positive. Positive operator A is

always bounded, i. e., its operator norm

$$\|A\| = \sup\{\|Ax\|_L : x \in L, \|x\|_L \leq 1\} = \sup\{\|Ax\|_L : x \in L_+, \|x\|_L \leq 1\} \quad (1.1)$$

is finite. For the proof see [33]. Its spectral radius $\rho(A)$ is always contained in the spectrum.

In the special case $L = L^2(X, \mu)$ we can define the numerical radius $w(A)$ of a bounded operator A on $L^2(X, \mu)$ by

$$w(A) = \sup\{|\langle Af, f \rangle| : f \in L^2(X, \mu), \|f\|_2 = 1\}.$$

If, in addition, A is positive, then

$$w(A) = \sup\{\langle Af, f \rangle : f \in L^2(X, \mu)_+, \|f\|_2 = 1\}.$$

From this it follows easily that $w(A) \leq w(B)$ for all positive operators A and B on $L^2(X, \mu)$ with $A \leq B$.

Definition 1.1.1 Let (X, μ) and (Y, ν) be σ -finite measure spaces, $L \subset M(Y, \nu)$ and $N \subset M(X, \mu)$ ideals. An operator $A : L \rightarrow N$ is called a kernel operator if there exists a $\mu \times \nu$ -measurable function $a(x, y)$ on $X \times Y$ such that Y is the carrier of N and for all $f \in L$ and for almost all $x \in X$,

$$\int_Y |a(x, y)f(y)| d\nu(y) < \infty \quad \text{and} \quad (Af)(x) = \int_Y a(x, y)f(y) d\nu(y).$$

A kernel operator A is positive iff its kernel a is non-negative almost everywhere. For the proof see e.g. [33, Izrek 1.15].

Let L be a Banach function space such that L and L^* have order continuous norms and let A and B be positive kernel operators on L . By $\gamma(A)$ we denote the Hausdorff measure of non-compactness of A , i.e.,

$$\gamma(A) = \inf \{\delta > 0 : \text{there is a finite } M \subset L \text{ such that } A(D_L) \subset M + \delta D_L\},$$

where $D_L = \{f \in L : \|f\|_L \leq 1\}$. Then $\gamma(A) \leq \|A\|$, $\gamma(A + B) \leq \gamma(A) + \gamma(B)$, $\gamma(AB) \leq \gamma(A)\gamma(B)$ and $\gamma(\alpha A) = \alpha\gamma(A)$ for $\alpha \geq 0$. Also $0 \leq A \leq B$ implies $\gamma(A) \leq \gamma(B)$ (see e.g. [30, Corollary 4.3.7 and Corollary 3.7.3]). Let $\rho_{ess}(A)$ denote the essential spectral radius of A , i.e., the spectral radius of the Calkin image of A in the Calkin algebra. Then

$$\rho_{ess}(A) = \lim_{j \rightarrow \infty} \gamma(A^j)^{1/j} = \inf_{j \in \mathbb{N}} \gamma(A^j)^{1/j} \quad (1.2)$$

and $\rho_{ess}(A) \leq \gamma(A)$. If $L = L^2(X, \mu)$, then $\gamma(A^*) = \gamma(A)$ and $\rho_{ess}(A^*) = \rho_{ess}(A)$, where A^* denotes the adjoint of A . Equalities (1.2) and $\rho_{ess}(A^*) = \rho_{ess}(A)$ are valid for any bounded operator A on a given complex Banach space L (see e.g. [30, Theorem 4.3.13 and Proposition 4.3.11]).

Let A and B be positive kernel operators on a Banach function space L with kernels a and b respectively, and $\alpha \geq 0$. The *Hadamard (or Schur) product* $A \circ B$ of A and B is the kernel operator with kernel equal to $a(x, y)b(x, y)$ at point $(x, y) \in X \times X$ which can be defined (in general) only on some order ideal of L . Similarly, the *Hadamard (or Schur) power* $A^{(\alpha)}$ of A is the kernel operator with kernel equal to $(a(x, y))^\alpha$ at point $(x, y) \in X \times X$ which can be defined only on some order ideal of L .

Let $A_1, \dots, A_m$ be positive kernel operators on a Banach function space L , and $\alpha_1, \dots, \alpha_m$ positive numbers such that $\sum_{j=1}^m \alpha_j = 1$. Then the *Hadamard weighted geometric mean* $A = A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \dots \circ A_m^{(\alpha_m)}$ of the operators $A_1, \dots, A_m$ is a positive kernel operator defined on the whole space L , since $A \leq \alpha_1 A_1 + \alpha_2 A_2 + \dots + \alpha_m A_m$ by the inequality between the weighted arithmetic and geometric means.

A matrix $A = [a_{ij}]_{i,j \in R}$ is called *nonnegative* if $a_{ij} \geq 0$ for all $i, j \in R$. For notational convenience, we sometimes write $a(i, j)$ instead of a_{ij} . We say that a nonnegative matrix A defines an operator on L if $Ax \in L$ for all $x \in L$, where $(Ax)_i = \sum_{j \in R} a_{ij}x_j$. Then $Ax \in L_+$ for all $x \in L_+$ and so A defines a positive kernel operator on L .

The problem of comparing the spectral radius $\rho(A \circ B)$ of the Schur product of two nonnegative matrices A and B in terms of $\rho(A \circ A)$ and $\rho(B \circ B)$ was motivated by studies of word relationships between random sequences generated from a m -letter alphabet $\mathcal{A}$ where the successive letters in each sequence occur as independent realizations of an m -state Markov chain with the transition matrix P (see e.g. [25]). Let $\mathcal{S}$ be such a randomly generated sequence with total length N . The length of the longest word occurring at least r times in $\mathcal{S}$ is denoted by $L_r^{(N)}$, where by a word in $\mathcal{S}$ of length k we mean a contiguous set of k letters from $\mathcal{A}$. Karlin and Ost established in [26] that the expected length of $L_r^{(N)}$ is of asymptotic order

$$\frac{\binom{N}{r}}{(-\log \rho_r)},$$

where $\rho_r = \rho(P^{(r)})$ is the Schur power of the transition matrix P .

Another motivation that leads to Schur products comes from comparing between several letter sequences. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two random strings of N letters from the alphabet $\mathcal{A}$, where the realization of $\mathcal{S}_1$ is governed by the Markov transition matrix P and the realization of $\mathcal{S}_2$ is governed by the Markov transition matrix Q . It is assumed that $\mathcal{S}_1$ and $\mathcal{S}_2$ are generated independently. The length of the longest word common to $\mathcal{S}_1$ and $\mathcal{S}_2$ is denoted by $W_{P,Q}^N$. In particular, when $Q = P^*$ the transition matrix Q is exactly that of the time reserved Markov chain to P . Another important class of examples is the case when $Q = \Pi^{-1}P\Pi$, where Π is a fixed permutation matrix. It was established by Karlin and Ost that the random variable $W_{P,Q}^N$ grows on the average as $(\log N^2)/(-\log \rho_{P,Q})$ where $\rho_{P,Q} = \rho(P \circ Q)$.

In [25] Karlin and Ost showed for the spectral radius of the Schur powers $\rho_r = \rho(A^{(r)})$ that the function $(1/r) \log \rho_r$ is strictly decreasing for $r > 0$ when A is a nonnegative irreducible matrix, while in the case when A is in addition a stochastic irreducible matrix the function $(1/r-1) \log \rho_r$ is strictly increasing for integers $r \geq 2$. In [25] they also established a connection between the entropy of a finite Markov chain and $d\rho_r/dr|_{r=1}$ stating that

$$\left. \frac{d\rho_r}{dr} \right|_{r=1} = H(\{X_n\})$$

where H is the entropy of the Markov chain $\{X_n\}$ associated with an irreducible matrix P . See e. g. [27, Chapter 9] for the definition of the entropy of a stationary Markov chain and the theory of the stationary processes.

1.2 Spectral radius of measurable functions

In this sections we present the results from [34] which enabled the proof of the key theorems Theorem 1.2.16 and Theorem 1.2.17 which are used to obtain further results.

First we consider measurable functions on the product measure space $X \times X$.

Definition 1.2.1 *Let $\phi : M(X, \mu)_+ \rightarrow [0, \infty]$ be a function norm such that for all $f, g \in M(X \times X)_+$ the function*

$$(f * g)(x, y) = \phi(f(x, \cdot)g(\cdot, y)) \tag{1.3}$$

is measurable, i.e., $f * g \in M(X \times X)_+$. By $f^{[1]} = f$, $f^{[j]} = f^{[j-1]} * f$ we define the powers of $f \in M(X \times X)_+$ with respect to $*$. We will also use the notation $f_1 * f_2 * \dots * f_m := (\dots (f_1 * f_2) * \dots * f_{m-1}) * f_m$ for $f_j \in M(X \times X)_+$.

Remark 1.2.2 In Definition 1.2.1 it is required that $f * g \in M(X \times X)_+$ for all $f, g \in M(X \times X)_+$. This is often satisfied according to Luxemburg-Gribanov theorem ([45, Theorem 99.2]). See also [34, Example 2.6] and the rest of the section below.

Definition 1.2.3 Let $h : M(X \times X)_+ \rightarrow [0, \infty]$ be a function seminorm. By

$$r_h(f) = \limsup_{j \rightarrow \infty} h(f^{[j]})^{1/j} \quad (1.4)$$

we define the spectral radius of $f \in M(X \times X)_+$ with respect to h and $*$.

The spectral radius $r_h : M(X \times X)_+ \rightarrow [0, \infty]$ is monotone and positively homogenous. If h is submultiplicative with respect to $*$ (i.e., $h(f * g) \leq h(f)h(g)$ for all $f, g \in M(X \times X)_+$), then $r_h(f) \leq h(f)$ for all $f \in M(X \times X)_+$.

The following result was proved in [34].

Theorem 1.2.4 Let $f, g \in M(X \times X)_+$ and let a function seminorm $h : M(X \times X)_+ \rightarrow [0, \infty]$ be submultiplicative with respect to the product $*$, which is associative. Then the following properties hold.

(i) If $f \in C_h$, then

$$r_h(f) = \inf_{j \in \mathbb{N}} h(f^{[j]})^{1/j} = \lim_{j \rightarrow \infty} h(f^{[j]})^{1/j}.$$

(ii) If $g * f \leq f * g$ or $f * g \leq g * f$, then

$$r_h(f * g) \leq r_h(f)r_h(g).$$

(iii) If $f, g \in C_h$ and $g * f \leq f * g$, then

$$r_h(f + g) \leq r_h(f) + r_h(g).$$

Proof: For the proof see [34, Proposition 2.4]. ■

Example 1.2.5 Let $\phi(u) = \int_X u(z)dz$ for $u \in M(X, \mu)_+$, then $*$ is convolution, i. e.,

$$(f * g)(x, y) = \int_X f(x, z)g(z, y) dz$$

for $f, g \in M(X \times X)_+$.

So, the spectral radius of a positive kernel operator is also a special case of $r_h(f)$. See Example 1.2.10 below.

Next, we will use the Young's inequality

$$x^\alpha y^{1-\alpha} \leq \alpha x + (1 - \alpha)y$$

for $x, y \geq 0$ and $\alpha \in (0, 1)$ and its sharpened version

$$x^\alpha y^{1-\alpha} = \inf_{t>0} \left\{ \alpha t^{\frac{1}{\alpha}} x + (1 - \alpha)t^{-\frac{1}{1-\alpha}} y \right\}. \quad (1.5)$$

The use of the sharpened version of the Young's inequality was proposed by Professor T. Ando to obtain alternative proof of some results from [10] in a letter addressed to Professor R. Drnovšek shortly after Positivity conference in Dresden. A. Peperko applied that in [34] after R. Drnovšek presented him the content of the letter.

The following proposition has been known for years (see e.g. [15, Proposition 1.1 and Remark 1.2.5]).

Proposition 1.2.6 Let $h : M(X, \mu)_+ \rightarrow [0, \infty]$ be a function seminorm and $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$. If $f_i \in C_h$ for $i = 1, \dots, m$, then $f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m} \in C_h$ and

$$h(f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m}) \leq h(f_1)^{\alpha_1} h(f_2)^{\alpha_2} \dots h(f_m)^{\alpha_m}. \quad (1.6)$$

If, in addition, h is a function norm, then (1.6) holds for arbitrary $f_i \in M(X, \mu)_+$, $i = 1, \dots, m$.

Proof: It is sufficient to prove (1.6) for $m = 2$. The rest of the proof follows by induction.

Let $f, g \in C_h$ and $\alpha \in (0, 1)$. We must show that

$$h(f^\alpha g^{1-\alpha}) \leq h(f)^\alpha h(g)^{1-\alpha}. \quad (1.7)$$

From Young's inequality, monotonicity and convexity of h we get

$$h(f^\alpha g^{1-\alpha}) \leq h(\alpha f + (1-\alpha)g) \leq \alpha h(f) + (1-\alpha)h(g). \quad (1.8)$$

Replacing f by $t^{\frac{1}{\alpha}} f$ and g by $t^{-\frac{1}{1-\alpha}} g$ for $t > 0$ in (1.8) we have

$$h(f^\alpha g^{1-\alpha}) \leq \alpha t^{\frac{1}{\alpha}} h(f) + (1-\alpha)t^{-\frac{1}{1-\alpha}} h(g).$$

Taking the infimum over all $t > 0$ we obtain the inequality (1.7). ■

Let $L \subset M(X, \mu)$ be an ideal, equipped with a lattice seminorm ρ . Since $\rho(f) = \rho(|f|)$ for all $f \in L$, the following result follows from Proposition 1.2.6 and was also presented in [34].

Corollary 1.2.7 *Let $L \subset M(X, \mu)$ be an ideal, equipped with a lattice seminorm ρ . Then $f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m} \in L$ and*

$$\rho(f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}) \leq \rho(f_1)^{\alpha_1} \rho(f_2)^{\alpha_2} \cdots \rho(f_m)^{\alpha_m} \quad (1.9)$$

for all $f_i \in L$ and $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$.

The following Lemma, Theorem and Examples listed below were also part of [34].

Lemma 1.2.8 *Let $f_i, g_i \in M(X \times X)_+$ and $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$. Then*

$$(f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}) * (g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_m^{\alpha_m}) \leq (f_1 * g_1)^{\alpha_1} (f_2 * g_2)^{\alpha_2} \cdots (f_m * g_m)^{\alpha_m} \quad (1.10)$$

and

$$(f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m})^{[j]} \leq (f_1^{[j]})^{\alpha_1} (f_2^{[j]})^{\alpha_2} \cdots (f_m^{[j]})^{\alpha_m} \quad (1.11)$$

for all $j \in \mathbb{N}$.

Proof: For almost all $(x, y) \in X \times X$ we have by Proposition 1.2.6 used for ϕ

$$((f_1^{\alpha_1} \cdots f_m^{\alpha_m}) * (g_1^{\alpha_1} \cdots g_m^{\alpha_m}))(x, y)$$

$$\begin{aligned}
 &= \phi(f_1^{\alpha_1}(x, \cdot) \cdots f_m^{\alpha_m}(x, \cdot) g_1^{\alpha_1}(\cdot, y) \cdots g_m^{\alpha_m}(\cdot, y)) \\
 &= \phi((f_1(x, \cdot) g_1(\cdot, y))^{\alpha_1} \cdots (f_m(x, \cdot) g_m(\cdot, y))^{\alpha_m}) \\
 &\leq \phi(f_1(x, \cdot) g_1(\cdot, y))^{\alpha_1} \cdots \phi(f_m(x, \cdot) g_m(\cdot, y))^{\alpha_m} \\
 &= (f_1 * g_1)^{\alpha_1}(x, y) \cdots (f_m * g_m)^{\alpha_m}(x, y),
 \end{aligned}$$

which proves (1.10).

Since ϕ is monotone, (1.11) follows from (1.10) by induction. ■

Theorem 1.2.9 *Let $h : M(X \times X)_+ \rightarrow [0, \infty]$ be a function seminorm and $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$. Then $(f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m})^{[j]} \in C_h$ and*

$$r_h(f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}) \leq r_h(f_1)^{\alpha_1} r_h(f_2)^{\alpha_2} \cdots r_h(f_m)^{\alpha_m} \quad (1.12)$$

for all $f_i \in M(X \times X)_+$ such that $f_i^{[j]} \in C_h$ for all $i = 1, \dots, m$ and $j \in \mathbb{N}$.

If, in addition, h is a function norm, then (1.12) holds for arbitrary $f_1, \dots, f_m \in M(X \times X)_+$.

Proof: Let $j \in \mathbb{N}$. From (1.11) we get

$$\begin{aligned}
 h\left((f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m})^{[j]}\right) &\leq h\left(\left(f_1^{[j]}\right)^{\alpha_1} \left(f_2^{[j]}\right)^{\alpha_2} \cdots \left(f_m^{[j]}\right)^{\alpha_m}\right) \\
 &\leq h\left(f_1^{[j]}\right)^{\alpha_1} h\left(f_2^{[j]}\right)^{\alpha_2} \cdots h\left(f_m^{[j]}\right)^{\alpha_m}
 \end{aligned}$$

by monotonicity of h and (1.6). Taking the j -th root and upper limits now gives (1.12). ■

Example 1.2.10 *Let (X, μ) and (Y, ν) be σ -finite measure spaces, L and N normed Köthe spaces in $M(Y, \nu)$ and $M(X, \mu)$ respectively, such that Y is the carrier of N . Assume that $\alpha_i > 0$, $i = 1, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$. Let C be a cone of all $a \in M(X \times Y)_+$ such that $a(x, y)$ is the kernel of a bounded positive kernel operator $A : L \rightarrow N$. The cone C is a solid subcone of $M(X \times Y)_+$. If we define $h(a) = \|A\|$ for $a \in C$ (where $\|A\|$ is the operator norm of operator A) and $h(a) = \infty$ for $a \notin C$, then $h : M(X \times Y)_+ \rightarrow [0, \infty]$ is a function norm and $C = C_h$.*

Let $A_1, \dots, A_m$ be bounded positive kernel operators from L into N with kernels $a_1, \dots, a_m$ respectively. Then we have by Proposition 1.2.6

$$\|A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}\| \leq \|A_1\|^{\alpha_1} \|A_2\|^{\alpha_2} \cdots \|A_m\|^{\alpha_m}, \quad (1.13)$$

since $\|A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \dots \circ A_m^{(\alpha_m)}\| = h(a_1^{\alpha_1} a_2^{\alpha_2} \dots a_m^{\alpha_m})$.

Let $(X, \mu) = (Y, \nu)$, $L = N$ and let ϕ be as in Example 1.2.5. Then $(a_1 * a_2)(x, y) = \int_X a_1(x, z) a_2(z, y) dz$ is the kernel of a bounded positive kernel operator $A_1 A_2$ on L . Therefore $a^{[j]}$ is the kernel of A^j for all $j \in \mathbb{N}$ and so $r_h(a) = r(A)$ (the spectral radius of the operator A). By Theorem 1.2.9 we have

$$r(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \dots \circ A_m^{(\alpha_m)}) \leq r(A_1)^{\alpha_1} r(A_2)^{\alpha_2} \dots r(A_m)^{\alpha_m}. \quad (1.14)$$

Example 1.2.11 Let L be a Banach function space in $M(X, \mu)$, such that X is the carrier of L and let L and L^* have order continuous norms. Let ϕ be as in Example 1.2.5 and $\alpha_i > 0$ for $i = 1, \dots, m$ such that $\sum_{i=1}^m \alpha_i = 1$ and let C be as in Example 1.2.10. Define $h(a) = \gamma(A)$ for $a \in C$ and $h(a) = \infty$ for $a \notin C$. Here $\gamma(A)$ denotes Hausdorff measure of non-compactness. Then $h : M(X \times X)_+ \rightarrow [0, \infty]$ is a function seminorm (see e.g. [30, Corollary 4.3.7 and Corollary 3.7.3]), $C = C_h$ and for $a \in C$ we have

$$r_h(a) = \lim_{j \rightarrow \infty} \gamma(A^j)^{1/j} = \inf_{j \in \mathbb{N}} \gamma(A^j)^{1/j} = r_{\text{ess}}(A)$$

(the essential spectral radius of K ; see e.g. [30, Theorem 4.3.13]). By Theorem 1.2.9 we have

$$r_{\text{ess}}(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \dots \circ A_m^{(\alpha_m)}) \leq r_{\text{ess}}(A_1)^{\alpha_1} r_{\text{ess}}(A_2)^{\alpha_2} \dots r_{\text{ess}}(A_m)^{\alpha_m},$$

where $A_1, A_2, \dots, A_m$ are positive kernel operators on L .

Now, we restrict our attention to the completely atomic case and consider measure spaces (X, μ) where either $X = \mathbb{N}$ or $X = \{1, 2, \dots, N\}$ for some $N \in \mathbb{N}$.

Proposition 1.2.12 Let $h : M(X, \mu)_+ \rightarrow [0, \infty]$ be a function seminorm satisfying the condition that $h(\chi_{\{i\}}) \geq 1$ for all $i \in X$. Let $t \geq 1$, $f \in M(X, \mu)_+$ and $\alpha_k > 0$, $k = 1, \dots, m$, such that $s_m = \sum_{k=1}^m \alpha_k \geq 1$. Then the following properties hold:

- (i) If $f(i) < \infty$, then $f(i) \leq h(f)$.
- (ii) $h(f^t) \leq h(f)^t$.

(iii) If $f_1, \dots, f_m \in C_h$, then $f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m} \in C_h$ and

$$h(f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m}) \leq h(f_1)^{\alpha_1} h(f_2)^{\alpha_2} \dots h(f_m)^{\alpha_m}. \quad (1.15)$$

If, in addition h is a function norm, then (1.15) holds for any $f_1, \dots, f_m \in M(X, \mu)_+$.

Proof: Let $f(i) < \infty$. Since $f(i)\chi_{\{i\}} \leq f$ for all $i \in X$, we have $h(f) \geq h(f(i)\chi_{\{i\}}) = f(i)h(\chi_{\{i\}}) \geq f(i)$. This establishes (i).

To prove (ii) we may assume that $0 < h(f) < \infty$ because of (i). Furthermore, we may assume that $h(f) = 1$, since h is positively homogeneous. If $f(i) < \infty$, then $f(i) \leq h(f) = 1$ by (i). It follows that $f^t \leq f$, since $t \geq 1$. Therefore $h(f^t) \leq h(f) = 1$, which proves (ii).

The property (iii) follows from (ii) and (1.6). ■

Lemma 1.2.13 *Let $t \geq 1$, $f_1, \dots, f_n \in M(X \times X)_+$ and $\phi(\chi_{\{i\}}) \geq 1$ for all $i \in X$. Then*

$$f_1^t * \dots * f_n^t \leq (f_1 * \dots * f_n)^t. \quad (1.16)$$

Let $h : M(X \times X)_+ \rightarrow [0, \infty]$ be a function seminorm satisfying the condition that $h(\chi_{\{(i,j)\}}) \geq 1$ for all $(i, j) \in X \times X$. Then

$$h(f_1^t * \dots * f_n^t) \leq h(f_1 * \dots * f_n)^t, \quad (1.17)$$

and

$$r_h(f_1^t * \dots * f_n^t) \leq r_h(f_1 * \dots * f_n)^t. \quad (1.18)$$

Proof: First we prove (1.16) for $n = 2$. We need to show that

$$f^t * g^t \leq (f * g)^t \quad (1.19)$$

for all $f, g \in M(X \times X)_+$. Take $(i, j) \in X \times X$. Then we have

$$\begin{aligned} (f^t * g^t)(i, j) &= \phi(f^t(i, \cdot)g^t(\cdot, j)) = \phi((f(i, \cdot)g(\cdot, j))^t) \\ &\leq \phi(f(i, \cdot)g(\cdot, j))^t = (f * g)^t(i, j) \end{aligned}$$

by Proposition 1.2.12(ii) used for ϕ , so we have proved (1.19).

Now (1.16) follows from (1.19) by induction. The inequality (1.17) follows from (1.16), monotonicity of h and Proposition 1.2.12(ii).

For the proof of (1.18) we first show that

$$r_h(f^t) \leq r_h(f)^t. \quad (1.20)$$

for $f \in M(X \times X)_+$. By (1.17) we have

$$h((f^t)^{[j]}) \leq h(f^{[j]})^t \quad (1.21)$$

for all $j \in \mathbb{N}$. Taking the j -th root and upper limits we obtain (1.20), while (1.18) follows from (1.16) and (1.20), which completes the proof. ■

Theorem 1.2.14 *Let h and ϕ be as in Lemma 1.2.13 and $\alpha_k > 0$, $k = 1, \dots, m$, such that $s_m = \sum_{k=1}^m \alpha_k \geq 1$. Then $(f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m})^{[j]} \in C_h$ and*

$$r_h(f_1^{\alpha_1} f_2^{\alpha_2} \dots f_m^{\alpha_m}) \leq r_h(f_1)^{\alpha_1} r_h(f_2)^{\alpha_2} \dots r_h(f_m)^{\alpha_m} \quad (1.22)$$

provided $f_1^{[j]}, \dots, f_m^{[j]} \in C_h$ for all $j \in \mathbb{N}$.

If, in addition, h is a function norm, then (1.22) holds for arbitrary $f_1, \dots, f_m \in M(X \times X)_+$.

Proof: Using (1.20) and (1.12) the proof of (1.22) is similar to the proof of Proposition 1.2.12 (iii). ■

Now let $\alpha_i > 0$, $i = 1, 2, \dots, m$, such that $\sum_{i=1}^m \alpha_i = 1$. From (1.10) we can (applying induction by n) conclude that

$$(f_{11}^{\alpha_1} \dots f_{1m}^{\alpha_m}) * \dots * (f_{n1}^{\alpha_1} \dots f_{nm}^{\alpha_m}) \leq (f_{11} * \dots * f_{n1})^{\alpha_1} \dots (f_{1m} * \dots * f_{nm})^{\alpha_m}. \quad (1.23)$$

for all $f_{ji} \in M(X \times X)_+$, $j = 1, \dots, n$, $i = 1, \dots, m$.

Theorem 1.2.15 *Let $h : M(X \times X)_+ \rightarrow [0, \infty]$ be a function seminorm and $f_{ik} \in M(X \times X)_+$, $i = 1, \dots, n$, $k = 1, \dots, m$ and $\alpha_k > 0$, such that $\sum_{k=1}^m \alpha_k = 1$.*

*If $f_{1k} * \dots * f_{nk} \in C_h$ for all $k = 1, \dots, m$, then $(f_{11}^{\alpha_1} \dots f_{1m}^{\alpha_m}) * \dots * (f_{n1}^{\alpha_1} \dots f_{nm}^{\alpha_m}) \in C_h$ and*

$$h((f_{11}^{\alpha_1} \dots f_{1m}^{\alpha_m}) * \dots * (f_{n1}^{\alpha_1} \dots f_{nm}^{\alpha_m})) \leq h(f_{11} * \dots * f_{n1})^{\alpha_1} \dots h(f_{1m} * \dots * f_{nm})^{\alpha_m} \quad (1.24)$$

and if $(f_{1k} * \dots * f_{nk})^{[j]} \in C_h$ for all $k = 1, \dots, m$ and $j \in \mathbb{N}$, then $((f_{11}^{\alpha_1} \dots f_{1m}^{\alpha_m}) * \dots * (f_{n1}^{\alpha_1} \dots f_{nm}^{\alpha_m}))^{[j]} \in C_h$ and

$$\begin{aligned} & r_h((f_{11}^{\alpha_1} \dots f_{1m}^{\alpha_m}) * \dots * (f_{n1}^{\alpha_1} \dots f_{nm}^{\alpha_m})) \\ & \leq r_h(f_{11} * \dots * f_{n1})^{\alpha_1} \dots r_h(f_{1m} * \dots * f_{nm})^{\alpha_m}. \end{aligned} \quad (1.25)$$

If, in addition, h is a function norm, then the inequalities (1.24) and (1.25) hold for arbitrary $f_{ik} \in M(X \times X)_+$.

Proof: Inequalities (1.24) and (1.25) follow from (1.23), Proposition 1.2.6 and Theorem 1.2.9, since h and r_h are monotone. $\blacksquare$

Theorem 1.2.16 Let $\{A_{ij}\}_{i=1, j=1}^{k, m}$ be positive kernel operators on a Banach function space L and let $\alpha_1, \alpha_2, \dots, \alpha_m$ are positive numbers.

(i) If $\sum_{j=1}^m \alpha_j = 1$, then the positive kernel operator

$$A := \left(A_{11}^{(\alpha_1)} \circ \dots \circ A_{1m}^{(\alpha_m)} \right) \dots \left(A_{k1}^{(\alpha_1)} \circ \dots \circ A_{km}^{(\alpha_m)} \right) \quad (1.26)$$

satisfies the following inequalities

$$A \leq (A_{11} \dots A_{k1})^{(\alpha_1)} \circ \dots \circ (A_{1m} \dots A_{km})^{(\alpha_m)}, \quad (1.27)$$

$$\begin{aligned} \|A\| & \leq \| (A_{11} \dots A_{k1})^{(\alpha_1)} \circ \dots \circ (A_{1m} \dots A_{km})^{(\alpha_m)} \| \\ & \leq \|A_{11} \dots A_{k1}\|^{\alpha_1} \dots \|A_{1m} \dots A_{km}\|^{\alpha_m} \end{aligned} \quad (1.28)$$

$$\begin{aligned} \rho(A) & \leq \rho((A_{11} \dots A_{k1})^{(\alpha_1)} \circ \dots \circ (A_{1m} \dots A_{km})^{(\alpha_m)}) \\ & \leq \rho(A_{11} \dots A_{k1})^{\alpha_1} \dots \rho(A_{1m} \dots A_{km})^{\alpha_m}. \end{aligned} \quad (1.29)$$

If, in addition, L and L^* have order continuous norms, then

$$\begin{aligned} \gamma(A) & \leq \gamma((A_{11} \dots A_{k1})^{(\alpha_1)} \circ \dots \circ (A_{1m} \dots A_{km})^{(\alpha_m)}) \\ & \leq \gamma(A_{11} \dots A_{k1})^{\alpha_1} \dots \gamma(A_{1m} \dots A_{km})^{\alpha_m}, \end{aligned} \quad (1.30)$$

$$\begin{aligned} \rho_{ess}(A) & \leq \rho_{ess}((A_{11} \dots A_{k1})^{(\alpha_1)} \circ \dots \circ (A_{1m} \dots A_{km})^{(\alpha_m)}) \\ & \leq \rho_{ess}(A_{11} \dots A_{k1})^{\alpha_1} \dots \rho_{ess}(A_{1m} \dots A_{km})^{\alpha_m}. \end{aligned} \quad (1.31)$$

In special case when $L = L^2(X, \mu)$ then

$$\begin{aligned} w(A) &\leq w((A_{11} \cdots A_{k1})^{(\alpha_1)} \circ \cdots \circ (A_{1m} \cdots A_{km})^{(\alpha_m)}) \\ &\leq w(A_{11} \cdots A_{k1})^{\alpha_1} \cdots w(A_{1m} \cdots A_{km})^{\alpha_m}. \end{aligned} \quad (1.32)$$

(ii) If $L \in \mathcal{L}$, $\sum_{j=1}^m \alpha_j \geq 1$ and $\{A_{ij}\}_{i=1, j=1}^{k, m}$ are nonnegative matrices that define positive operators on L , then A from (1.26) defines a positive operator on L and the inequalities (1.27), (1.28) and (1.29) hold.

Proof: (i) The proof of (1.27), (1.28) and (1.29) follows from Theorem 1.2.15 and Example 1.2.10. To prove (1.32) notice that numerical radius w on $L^2(X, \mu)$ is monotone and that we can apply the proof of the Theorem 1.2.15.

(ii) The proof of the inequalities (1.27), (1.28) and (1.29) in that case is based on Proposition 1.2.12(ii), Theorem 1.2.14 and Example 1.2.10. ■

The following theorem is a special case of Theorem 1.2.16.

Theorem 1.2.17 *Let $A_1, \dots, A_m$ be positive kernel operators on a Banach function space L and $\alpha_1, \dots, \alpha_m$ positive numbers. (i) If $\sum_{j=1}^m \alpha_j = 1$, then*

$$\|A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}\| \leq \|A_1\|^{\alpha_1} \|A_2\|^{\alpha_2} \cdots \|A_m\|^{\alpha_m} \quad (1.33)$$

and

$$\rho(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \leq \rho(A_1)^{\alpha_1} \rho(A_2)^{\alpha_2} \cdots \rho(A_m)^{\alpha_m}. \quad (1.34)$$

If, in addition, L and L^* have order continuous norms, then

$$\gamma(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \leq \gamma(A_1)^{\alpha_1} \gamma(A_2)^{\alpha_2} \cdots \gamma(A_m)^{\alpha_m} \quad (1.35)$$

and

$$\rho_{ess}(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \leq \rho_{ess}(A_1)^{\alpha_1} \rho_{ess}(A_2)^{\alpha_2} \cdots \rho_{ess}(A_m)^{\alpha_m}. \quad (1.36)$$

If, in special case $L = L^2(X, \mu)$ then

$$w(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}) \leq w(A_1)^{\alpha_1} w(A_2)^{\alpha_2} \cdots w(A_m)^{\alpha_m} \quad (1.37)$$

(ii) If $L \in \mathcal{L}$, $\sum_{j=1}^m \alpha_j \geq 1$ and if $A_1, \dots, A_m$ are nonnegative matrices that define positive operators on L , then $A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_m^{(\alpha_m)}$ defines a positive operator on L and (1.33) and (1.34) hold.

(iii) If $L \in \mathcal{L}$, $t \geq 1$ and if $A, A_1, \dots, A_m$ are nonnegative matrices that define operators on L , then $A^{(t)}$ defines an operator on L and the following inequalities hold

$$A_1^{(t)} \cdots A_m^{(t)} \leq (A_1 \cdots A_m)^{(t)}, \quad (1.38)$$

$$\rho(A_1^{(t)} \cdots A_m^{(t)}) \leq \rho(A_1 \cdots A_m)^t, \quad (1.39)$$

$$\|A_1^{(t)} \cdots A_m^{(t)}\| \leq \|A_1 \cdots A_m\|^t. \quad (1.40)$$

1.3 Essential spectral radius of Schur-Hadamard weighted geometric means

In this section we present some results from [28], including the essential versions of Theorems 1.2.16(ii), 1.2.17(ii)-(iii), under the assumption that L and L^* have order continuous norms.

We start with the following lemma, proved in [28].

Lemma 1.3.1 *Let $L \in \mathcal{L}$ have order continuous norm. Then for each $x \in L$ it holds that $x(i) \rightarrow 0$ as $i \rightarrow \infty$.*

Proof: Suppose there exists $x \in L$ such that the entries $x(i)$ do not converge to zero as $i \rightarrow \infty$. Then there exists $\varepsilon > 0$ such that there are infinitely many positive entries of $|x|$ that are greater than ε . For $k \in \mathbb{N}$ let $x_k(i) = 0$ when $i \leq k$ and $x_k(i) = |x|(i)$ otherwise. Then $0 \leq x_k \downarrow 0$. However, $\|x_k\|$ does not converge to zero, since we have $\|x_k\| \geq \| |x|(i) \cdot e_i \| = |x(i)| > \varepsilon$ for infinitely many $i > k$. ■

For the proof of the Theorem 1.3.3, given in [28] we need the following result, proved in [39, Corollary 2.10].

Theorem 1.3.2

$$A^{(t)} \leq \|A\|_\infty^{t-1} A, \quad (1.41)$$

$$\|A^{(t)}\| \leq \|A\|_\infty^{t-1} \|A\|, \quad (1.42)$$

$$\rho(A^{(t)}) \leq \|A\|_\infty^{t-1} \rho(A). \quad (1.43)$$

If, in addition, L and L^* have order continuous norms, then

$$\gamma(A^{(t)}) \leq \|A\|_\infty^{t-1} \gamma(A), \quad (1.44)$$

$$\rho_{ess}(A^{(t)}) \leq \|A\|_\infty^{t-1} \rho_{ess}(A). \quad (1.45)$$

First we establish the essential version of Theorem 1.2.17(iii), that was proved in [28].

Theorem 1.3.3 *Let $L \in \mathcal{L}$ such that L and L^* have order continuous norms. Let $t \geq 1$ and let $A, A_1, \dots, A_m$ be nonnegative matrices that define operators on L . Then*

$$\gamma(A^{(t)}) \leq \gamma(A)^t, \quad (1.46)$$

$$\rho_{ess}(A^{(t)}) \leq \rho_{ess}(A)^t, \quad (1.47)$$

$$\gamma(A_1^{(t)} \cdots A_m^{(t)}) \leq \gamma(A_1 \cdots A_m)^t, \quad (1.48)$$

$$\rho_{ess}(A_1^{(t)} \cdots A_m^{(t)}) \leq \rho_{ess}(A_1 \cdots A_m)^t. \quad (1.49)$$

Proof: First we prove (1.46). If $\gamma(A) = 0$, then $\gamma(A^{(t)}) = 0$ by (1.44). We may assume that $t > 1$. We may also assume that $\gamma(A) = 1$ since $\gamma(\cdot)$ is positively homogeneous. Having $\gamma(A) = 1$ means that for any $\delta > 1$, there is a finite set $U \subset L$ such that $A(D_L) \subset \bigcup_{u \in U} (u + \delta D_L)$, where $A(D_L)$ is the image of the closed unit ball D_L . Since U is a finite set in L , then by Lemma 1.3.1 there are only finitely many entries i such that $\max_{u \in U} |u_i| > \delta^2 - \delta$. Let I denote this set of indices. For all other indices $i \notin I$, we must have $(Ax)_i \leq \max |u_i| + \delta \leq \delta^2$ for all $x \in D_L$, $x \geq 0$. In particular, $A_{ij} = (Ae_j)_i \leq \delta^2$ for all j and all $i \notin I$.

Then $\delta^{-2t} A_{ij}^t \leq A_{ij}$ for all $i \notin I$, $j \in \mathbb{N}$ and $t > 1$. This means that $\delta^{-2t} A_i^{(t)} \leq A_i$ for all rows A_i such that $i \notin I$. Let P_I be the orthogonal projection onto $\text{span}\{e_i : i \in I\}$. Then $P_I A^{(t)}$ is compact since it has finite dimensional range, and if $Q_I = \text{id} - P_I$, then $\delta^{-2t} Q_I A^{(t)} \leq A$ and $\delta^{-2t} \gamma(A^{(t)}) = \delta^{-2t} \gamma(Q_I A^{(t)}) \leq \gamma(A) = 1$ (since $\gamma(\cdot)$ is invariant under compact perturbations and since it is monotone). Then $\gamma(A^{(t)}) \leq \delta^{2t}$. Since

$\delta > 1$ can be chosen arbitrarily close to 1, we conclude that $\gamma(A^{(t)}) \leq 1$ for all $t > 1$. This proves (1.46).

Inequality (1.48) follows from (1.38), monotonicity of $\gamma(\cdot)$ and (1.46). Inequality (1.49) follows from (1.2) and (1.48) since

$$\rho_{ess}(A) = \lim_{j \rightarrow \infty} \gamma((A_1^{(t)} \cdots A_m^{(t)})^j)^{1/j} \leq \lim_{j \rightarrow \infty} \gamma((A_1 \cdots A_m)^j)^{t/j} = \rho_{ess}(A_1 \cdots A_m)^t.$$

Inequality (1.47) is a special case of (1.49). ■

The following essential versions of Theorems 1.2.16(ii) and 1.2.17(ii) were established in [28] applying standard techniques used also in [10] and [34].

Theorem 1.3.4 *Let $L \in \mathcal{L}$ such that L and L^* have order continuous norms. Assume $A_1, \dots, A_m$ are nonnegative matrices that define operators on L and let $\alpha_1, \dots, \alpha_m$ be positive numbers such that $s_m = \sum_{j=1}^m \alpha_j \geq 1$. Then inequalities (1.35) and (1.36) hold.*

Proof: For $j = 1, \dots, m$ define $\beta_j = \frac{\alpha_j}{s_m}$ and so $\sum_{j=1}^m \beta_j = 1$. Then by (1.46) and Theorem 1.2.17(i) we have

$$\begin{aligned} \gamma(A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)}) &= \gamma\left(\left(A_1^{(\beta_1)} \circ \cdots \circ A_m^{(\beta_m)}\right)^{(s_m)}\right) \leq \gamma\left(A_1^{(\beta_1)} \circ \cdots \circ A_m^{(\beta_m)}\right)^{s_m} \\ &\leq (\gamma(A_1)^{\beta_1} \cdots \gamma(A_m)^{\beta_m})^{s_m} = \gamma(A_1)^{\alpha_1} \gamma(A_2)^{\alpha_2} \cdots \gamma(A_m)^{\alpha_m}, \end{aligned}$$

which proves (1.35) under our assumptions. Similarly, (1.36) follows from (1.47) and Theorem 1.2.17(i). ■

Theorem 1.3.5 *Let $L \in \mathcal{L}$ such that L and L^* have order continuous norms. Assume $\{A_{ij}\}_{i=1, j=1}^{k, m}$ are nonnegative matrices that define operators on L and let $\alpha_1, \dots, \alpha_m$ be positive numbers such that $s_m = \sum_{j=1}^m \alpha_j \geq 1$. Then for A from (1.26) inequalities (1.30) and (1.31) hold.*

Proof: Inequalities (1.30) and (1.31) under our assumptions follow from (1.27) in Theorem 1.2.16(ii), monotonicity of $\gamma(\cdot)$ and $\rho_{ess}(\cdot)$ and from Theorem 1.3.4. ■

Now we turn our attention to hyponormal operators. A bounded linear operator on A on a Hilbert space $\mathcal{H}$ is called *hyponormal* if $A^*A \geq AA^*$, that is $\|Ax\| \geq \|A^*x\|$ for all $x \in \mathcal{H}$, or equivalently if $A^*A - AA^*$ is positive

semidefinite. In particular, every normal operator is hyponormal. Let $\mathcal{B}(\mathcal{H})$ denote the Banach algebra of bounded linear operators on $\mathcal{H}$ and let π be the canonical projection of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{C}_\infty(\mathcal{H})$, where $\mathcal{C}_\infty(\mathcal{H})$ is the set of all compact operators on $\mathcal{H}$. Since the set $\mathcal{C}_\infty(\mathcal{H})$ is a closed two-sided ideal in $\mathcal{B}(\mathcal{H})$, the Calkin algebra is a C^* -algebra and the canonical projection is a $*$ -isomorphism. The essential norm of $A \in \mathcal{B}(\mathcal{H})$ is by definition $\|A\|_{ess} = \|\pi(A)\|$ and we have $\rho_{ess}(A) = \rho(\pi(A))$.

The following proposition and the following lemma were proved in [28]. For more background information see e. g. [32] and [43].

Proposition 1.3.6 *Let $\mathcal{H}$ be a Hilbert space. If $A \in \mathcal{B}(\mathcal{H})$ is hyponormal, then*

$$\rho_{ess}(A) = \gamma(A) = \|A\|_{ess}.$$

Proof: For $A \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{C}_\infty(\mathcal{H})$ it is clear that $\gamma(A) = \gamma(A + K) \leq \|A + K\|$. Therefore $\gamma(A) \leq \|A\|_{ess}$. By (1.2) [32, Theorem 1] and since $\gamma(A^n) \leq \gamma(A)^n$, it follows that

$$\rho_{ess}(A) \leq \gamma(A) \leq \|A\|_{ess}.$$

It remains to show that $\rho_{ess}(A) = \|A\|_{ess}$ when A is hyponormal. Since the spectrum of $\pi(A^*A - AA^*)$ is a subset of the spectrum of $A^*A - AA^*$ (see e.g [12, Theorem 2.3]), it follows that $\pi(A^*A - AA^*)$ is positive and therefore $\pi(A)$ is hyponormal whenever A is hyponormal. In that case, [43, Theorem 1] says that $\rho(\pi(A)) = \|\pi(A)\|$ and $\rho_{ess}(A) = \|A\|_{ess}$. ■

Lemma 1.3.7 *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$. Then $\rho_{ess}(A^*A) = \gamma(A^*A) = \gamma(A)^2$. Consequently Equalities below hold*

$$\rho_{ess}(A^*A) = \rho_{ess}(AA^*) = \gamma(A^*A) = \gamma(AA^*) = \gamma(A)^2 \quad (1.50)$$

and $\gamma(A) = \gamma(A^*)$.

Proof: By the polar decomposition theorem for bounded operators on a Hilbert space, $A = VD$ where V is a partial isometry and $D = \sqrt{A^*A}$. It follows that $\rho_{ess}(A^*A) = \rho_{ess}(D^2) = \rho_{ess}(D)^2$. By Proposition 1.3.6, $\rho_{ess}(D) = \gamma(D)$. Since V is a partial isometry, $\gamma(V) \leq \|V\| \leq 1$ and we have

$$\gamma(A)^2 = \gamma(VD)^2 \leq \gamma(D)^2 = \rho_{ess}(D)^2 = \rho_{ess}(A^*A). \quad (1.51)$$

It remains to prove the reverse inequality. Since $\gamma(V^*) \leq \|V^*\| = \|V\| \leq 1$, we have

$$\gamma(A^*A) = \gamma(DV^*A) \leq \gamma(D)\gamma(A).$$

Since $\rho_{ess}(A^*A) = \gamma(A^*A) = \gamma(D)^2$, we conclude that $\rho_{ess}(A^*A) \leq \gamma(A)^2$, which together with (1.51) establishes $\rho_{ess}(A^*A) = \gamma(A^*A) = \gamma(A)^2$. By (1.62) and Proposition 1.3.6 also the remaining equalities in (1.50) follow. The equality $\gamma(T) = \gamma(T^*)$ follows from (1.50). $\blacksquare$

Let $A_1 = [a_1(i, j)]_{i, j \in R}, \dots, A_n = [a_n(i, j)]_{i, j \in R}$ be nonnegative matrices and let $\alpha_1, \dots, \alpha_n$ be nonnegative numbers such that $\sum_{i=1}^n \alpha_i = 1$. The nonnegative matrix $C(A_1, \dots, A_n, \alpha_1, \dots, \alpha_n) = [c(i, j)]_{i, j \in R}$ is defined by

$$c(i, j) = \begin{cases} a_1^{\alpha_1}(i, j) \cdots a_n^{\alpha_n}(i, j) & \text{if } i \neq j \\ \alpha_1 a_1(i, i) + \cdots + \alpha_n a_n(i, i) & \text{if } i = j \end{cases}.$$

The diagonal part of $C(A_1, \dots, A_n, \alpha_1, \dots, \alpha_n)$ is equal to the diagonal part of $\alpha_1 A_1 + \cdots + \alpha_n A_n$, while its nondiagonal part equals the nondiagonal part of $A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_n^{(\alpha_n)}$.

By the inequality between weighted geometric and weighted arithmetic means, we have

$$A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \cdots \circ A_n^{(\alpha_n)} \leq C(A_1, \dots, A_n, \alpha_1, \dots, \alpha_n) \leq \alpha_1 A_1 + \cdots + \alpha_n A_n. \quad (1.52)$$

It follows that the matrix $C(A_1, \dots, A_n, \alpha_1, \dots, \alpha_n)$ defines an operator on L provided the matrices $A_1, \dots, A_n$ define operators on $L \in \mathcal{L}$. The following result was obtained in [11] and in [39].

Theorem 1.3.8 *Given L in $\mathcal{L}$, let $A_1, \dots, A_n$ be nonnegative matrices that define operators on L and $\alpha_1, \dots, \alpha_n$ nonnegative numbers such that $\sum_{i=1}^n \alpha_i = 1$. Then for $r = \rho$ we have*

$$r(C(A_1, \dots, A_n, \alpha_1, \dots, \alpha_n)) \leq \alpha_1 r(A_1) + \cdots + \alpha_n r(A_n). \quad (1.53)$$

In particular, if $A_1, \dots, A_n$ have the same non-diagonal part, then

$$r(\alpha_1 A_1 + \cdots + \alpha_n A_n) \leq \alpha_1 r(A_1) + \cdots + \alpha_n r(A_n). \quad (1.54)$$

In other words, if $D_1, \dots, D_n$ are diagonal matrices and A a matrix such that $A + D_1, \dots, A + D_n$ are nonnegative matrices that define operators on

L , then we have

$$r(\alpha_1(A + D_1) + \cdots + \alpha_n(A + D_n)) \leq \alpha_1 r(A + D_1) + \cdots + \alpha_n r(A + D_n). \quad (1.55)$$

If, in addition, L and L^* have order continuous norms then under the above conditions inequalities (1.53), (1.54) and (1.55) hold also for $r = \rho_{ess}$.

1.4 Inequalities for the joint and generalized spectral radius

The following section contains basic notions of the joint and the generalized spectral radius, as well as notion of Hadamard weighted geometric mean of sets, and some results established in [36] and [38].

We start with the following inequalities. For the proof see [31].

For nonnegative measurable functions and for nonnegative numbers α and β such that $\alpha + \beta \geq 1$ we have

$$f_1^\alpha g_1^\beta + \cdots + f_m^\alpha g_m^\beta \leq (f_1 + \cdots + f_m)^\alpha (g_1 + \cdots + g_m)^\beta \quad (1.56)$$

More generally, for nonnegative measurable functions $\{f_{ij}\}_{i=1, j=1}^{n, m}$ and for nonnegative numbers α_j , $j = 1, \dots, m$, such that $\sum_{j=1}^m \alpha_j \geq 1$ we have

$$(f_{11}^{\alpha_1} \cdots f_{1m}^{\alpha_m}) + \cdots + (f_{n1}^{\alpha_1} \cdots f_{nm}^{\alpha_m}) \leq (f_{11} + \cdots + f_{n1})^{\alpha_1} \cdots (f_{1m} + \cdots + f_{nm})^{\alpha_m} \quad (1.57)$$

Let Σ be a bounded set of bounded operators on a complex Banach space L . For $m \geq 1$, let

$$\Sigma^m = \{A_1 A_2 \cdots A_m : A_i \in \Sigma\}.$$

The generalized spectral radius of Σ is defined by

$$\rho(\Sigma) = \limsup_{m \rightarrow \infty} \left[\sup_{A \in \Sigma^m} \rho(A) \right]^{1/m} \quad (1.58)$$

and is equal to

$$\rho(\Sigma) = \sup_{m \in \mathbb{N}} \left[\sup_{A \in \Sigma^m} \rho(A) \right]^{1/m}.$$

The joint spectral radius of Σ is defined by

$$\hat{\rho}(\Sigma) = \lim_{m \rightarrow \infty} [\sup_{A \in \Sigma^m} \|A\|]^{1/m}. \quad (1.59)$$

Similarly, the generalized essential spectral radius of Σ is defined by

$$\rho_{ess}(\Sigma) = \limsup_{m \rightarrow \infty} [\sup_{A \in \Sigma^m} \rho_{ess}(A)]^{1/m} \quad (1.60)$$

and is equal to

$$\rho_{ess}(\Sigma) = \sup_{m \in \mathbb{N}} [\sup_{A \in \Sigma^m} \rho_{ess}(A)]^{1/m}.$$

The joint essential spectral radius of Σ is defined by

$$\hat{\rho}_{ess}(\Sigma) = \lim_{m \rightarrow \infty} [\sup_{A \in \Sigma^m} \gamma(A)]^{1/m}. \quad (1.61)$$

We will use the following well known facts that hold for all $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$:

$$r(\Sigma^m) = r(\Sigma)^m \quad \text{and} \quad r(\Psi\Sigma) = r(\Sigma\Psi) \quad (1.62)$$

where $\Psi\Sigma = \{AB : A \in \Psi, B \in \Sigma\}$ and $m \in \mathbb{N}$. The sum of bounded sets Ψ and Σ is a bounded set defined by $\Psi + \Sigma = \{A + B : A \in \Psi, B \in \Sigma\}$. Let $\Psi_1, \dots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \dots, \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i = 1$. Then the bounded set of positive kernel operators on L , defined by

$$\Psi_1^{(\alpha_1)} \circ \dots \circ \Psi_m^{(\alpha_m)} = \{A_1^{(\alpha_1)} \circ \dots \circ A_m^{(\alpha_m)} : A_1 \in \Psi_1, \dots, A_m \in \Psi_m\},$$

is called the *weighted Hadamard (Schur) geometric mean* of sets $\Psi_1, \dots, \Psi_m$. The set $\Psi_1^{(\frac{1}{m})} \circ \dots \circ \Psi_m^{(\frac{1}{m})}$ is called the *Hadamard (Schur) geometric mean* of sets $\Psi_1, \dots, \Psi_m$.

The following result that follows from Theorem 1.2.16(i) was established in [36, Theorem 3.3] and [38, Theorems 3.1 and 3.8].

Theorem 1.4.1 *Let $\Psi_1, \dots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \dots, \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i = 1$. If $r \in \{\rho, \hat{\rho}\}$ and $n \in \mathbb{N}$, then*

$$r(\Psi_1^{(\alpha_1)} \circ \dots \circ \Psi_m^{(\alpha_m)}) \leq r((\Psi_1^n)^{(\alpha_1)} \circ \dots \circ (\Psi_m^n)^{(\alpha_m)})^{\frac{1}{n}} \leq r(\Psi_1)^{\alpha_1} \dots r(\Psi_m)^{\alpha_m} \quad (1.63)$$

and

$$r \left(\Psi_1^{(\frac{1}{m})} \circ \dots \circ \Psi_m^{(\frac{1}{m})} \right) \leq r(\Psi_1 \Psi_2 \dots \Psi_m)^{\frac{1}{m}}. \quad (1.64)$$

If, in addition, L and L^* have order continuous norms, then (1.63) and (1.64) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

The following result was established in [38, Theorem 3.5].

Theorem 1.4.2 *Let Ψ and Σ be bounded sets of positive kernel operators on a Banach function space L . If $r \in \{\rho, \hat{\rho}\}$ and $\beta \in [0, 1]$, then we have*

$$\begin{aligned} r \left(\Psi^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})} \right) &\leq r \left((\Psi \Sigma)^{(\frac{1}{2})} \circ (\Sigma \Psi)^{(\frac{1}{2})} \right)^{\frac{1}{2}} \\ &\leq r \left((\Psi \Sigma)^{(\frac{1}{2})} \circ (\Psi \Sigma)^{(\frac{1}{2})} \right)^{\frac{1}{4}} r \left((\Sigma \Psi)^{(\frac{1}{2})} \circ (\Sigma \Psi)^{(\frac{1}{2})} \right)^{\frac{1}{4}} \leq r(\Psi \Sigma)^{\frac{1}{2}}, \end{aligned} \quad (1.65)$$

$$\begin{aligned} r \left(\Psi^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})} \right) &\leq r \left(\left(\Psi^{(\frac{1}{2})} \circ \Psi^{(\frac{1}{2})} \right) \left(\Sigma^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})} \right) \right)^{\frac{1}{2}} \\ &\leq r \left((\Psi \Sigma)^{(\frac{1}{2})} \circ (\Psi \Sigma)^{(\frac{1}{2})} \right)^{\frac{\beta}{2}} r \left((\Sigma \Psi)^{(\frac{1}{2})} \circ (\Sigma \Psi)^{(\frac{1}{2})} \right)^{\frac{1-\beta}{2}} \leq r(\Psi \Sigma)^{\frac{1}{2}}. \end{aligned} \quad (1.66)$$

If, in addition, L and L^* have order continuous norms, then (1.65) and (1.66) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

Given $L \in \mathcal{L}$, let $\Psi_1, \dots, \Psi_m$ be bounded sets of nonnegative matrices that define operators on L and let $\alpha_1, \dots, \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i \geq 1$. Then the set

$$\Psi_1^{(\alpha_1)} \circ \dots \circ \Psi_m^{(\alpha_m)} = \{A_1^{(\alpha_1)} \circ \dots \circ A_m^{(\alpha_m)} : A_1 \in \Psi_1, \dots, A_m \in \Psi_m\}$$

is a bounded set of nonnegative matrices that define operators on L by Theorem 1.2.17(ii).

By applying Theorem 1.2.16(ii) the following result can be also proved in a similar way as in [38, Theorem 3.8].

Theorem 1.4.3 *Given $L \in \mathcal{L}$, let $\Psi, \Psi_1, \dots, \Psi_m$ be bounded sets of nonnegative matrices that define operators on L . Let $\alpha_1, \dots, \alpha_m$ be positive numbers such that $\sum_{j=1}^m \alpha_j \geq 1$, $n \in \mathbb{N}$ and $r \in \{\rho, \hat{\rho}\}$. Then Inequalities (1.63) hold. In particular, if $t \geq 1$, then*

$$r(\Psi^{(t)}) \leq r((\Psi^n)^{(t)})^{\frac{1}{n}} \leq r(\Psi)^t. \quad (1.67)$$

Chapter 2

Inequalities for Schur-Hadamard weighted geometric mean

2.1 Geometric symmetrization

In this section we present some results from [5] regarding geometric symmetrization.

Let A be a positive kernel operator on $L^2(X, \mu)$ with kernel a . The geometric symmetrization $S(A)$ of A is the positive selfadjoint kernel operator on $L^2(X, \mu)$ with kernel equal to $\sqrt{a(x, y)a(y, x)}$ at point $(x, y) \in X \times X$. $S(A) = A^{(1/2)} \circ (A^*)^{(1/2)}$, since the kernel of the adjoint operator A^* is equal to $a(y, x)$ at point $(x, y) \in X \times X$.

Let A be a positive kernel operator on $L^2(X, \mu)$ and $\alpha \in [0, 1]$. Denote $S_\alpha(A) = A^{(\alpha)} \circ (A^*)^{(1-\alpha)}$, which is a kernel operator on $L^2(X, \mu)$ with a kernel $a^\alpha(x, y)a^{1-\alpha}(y, x)$. We also have $(S_\alpha(A))^* = S_\alpha(A^*) = S_{1-\alpha}(A)$.

The following result generalizes and refines [39, Propositions 3.1 and 3.2] and was proved in [5].

Proposition 2.1.1 *Let $A, A_1, \dots, A_n$ be positive kernel operators on $L^2(X, \mu)$ and $\alpha \in [0, 1]$. Then we have*

$$r(S_\alpha(A_1) \cdots S_\alpha(A_n))$$

$$\leq r \left((A_1 \cdots A_n)^{(\alpha)} \circ ((A_n \cdots A_1)^*)^{(1-\alpha)} \right) \leq r(A_1 \cdots A_n)^\alpha r(A_n \cdots A_1)^{1-\alpha}, \quad (2.1)$$

$$r(S_\alpha(A_1) + \cdots + S_\alpha(A_m)) \leq r(S_\alpha(A_1 + \cdots + A_m)) \leq r(A_1 + \cdots + A_m) \quad (2.2)$$

for all $r \in \{\rho, \rho_{ess}, \gamma, \|\cdot\|, w\}$. In particular, for all $r \in \{\rho, \rho_{ess}, \gamma, \|\cdot\|, w\}$ we have

$$r(S_\alpha(A)) \leq r(A). \quad (2.3)$$

We also have

$$r(S_\alpha(A_1)S_\alpha(A_2)) \leq r((A_1A_2)^{(\alpha)} \circ ((A_2A_1)^*)^{(1-\alpha)}) \leq r(A_1A_2). \quad (2.4)$$

for $r \in \{\rho, \rho_{ess}\}$.

Proof: We have by (1.28), (1.29), (1.30) and (1.31).

$$\begin{aligned} r(S_\alpha(A_1) \cdots S_\alpha(A_n)) &= r \left((A_1^{(\alpha)} \circ (A_1^*)^{(1-\alpha)}) \cdots (A_n^{(\alpha)} \circ (A_n^*)^{(1-\alpha)}) \right) \\ &\leq r \left((A_1 \cdots A_n)^{(\alpha)} \circ ((A_n \cdots A_1)^*)^{(1-\alpha)} \right) \\ &\leq r(A_1 \cdots A_n)^\alpha \rho((A_n \cdots A_1)^*)^{1-\alpha} = r(A_1 \cdots A_n)^\alpha \rho(A_n \cdots A_1)^{1-\alpha}. \end{aligned}$$

This proves (2.1). In particular, the inequality (2.3) is a special case of (2.1) and (2.4) follows from (2.1) and $r(A_1A_2) = r(A_2A_1)$ for $r \in \{\rho, \rho_{ess}\}$. Inequalities (2.2) follow from (1.56) and (2.3). ■

If A is a nonnegative matrix that defines an operator on $l^2(R)$ and if α and β are nonnegative numbers such that $\alpha + \beta \geq 1$, then a nonnegative matrix $S_{\alpha,\beta}(A) = A^{(\alpha)} \circ (A^*)^{(\beta)}$ also defines an operator on $l^2(R)$ by Theorem 1.2.16 (ii).

The following result was established in [5] and generalizes [9, Lemma 2.1] and [9, Theorem 2.2].

Lemma 2.1.2 (i) *If A is a positive kernel operator on $L^2(X, \mu)$ and $\alpha \in [0, 1]$, then*

$$S_\alpha(A^2) \geq S_\alpha(A)^2. \quad (2.5)$$

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(ii) If A is a nonnegative matrix that defines an operator on $l^2(R)$ and if α and β are nonnegative numbers such that $\alpha + \beta \geq 1$, then

$$S_{\alpha,\beta}(A^2) \geq S_{\alpha,\beta}(A)^2. \quad (2.6)$$

Proof: The kernel of $S_\alpha(A^2)$ at $(x, y) \in X \times X$ is equal to

$$\left(\int_X a(x, z)a(z, y)d\mu(z) \right)^\alpha \left(\int_X a(y, z)a(z, x)d\mu(z) \right)^{1-\alpha}.$$

This is larger or equal to

$$\begin{aligned} & \int_X (a(x, z)a(z, y))^\alpha (a(y, z)a(z, x))^{1-\alpha} d\mu(z) \\ &= \int_X a(x, z)^\alpha a(z, x)^{1-\alpha} a(z, y)^\alpha a(y, z)^{1-\alpha} d\mu(z) \end{aligned}$$

and this equals the kernel of $S_\alpha(A)^2$ at (x, y) , which proves (2.5).

Inequality (2.6) is proved in a similar way by [34, Proposition 4.1]. ■

The following result was proved in [5] and generalizes [39, Theorem 3.5] and its method of proof.

Theorem 2.1.3 *Let A be a positive kernel operator on $L^2(X, \mu)$, $\alpha \in [0, 1]$ and let $r_n = r(S_\alpha(A^{2^n}))^{2^{-n}}$ for $n \in \mathbb{N} \cup \{0\}$ and $r \in \{\rho, \rho_{ess}\}$. Then for each n*

$$r(S_\alpha(A)) = r_0 \leq r_1 \leq \cdots \leq r_n \leq r(A).$$

Proof: By (2.5) we have

$$r(S_\alpha(A^2)) \geq r(S_\alpha(A)^2) = r(S_\alpha(A))^2. \quad (2.7)$$

Using (2.3) we obtain $r(S_\alpha(A^{2^n})) \leq r(A^{2^n}) = r(A)^{2^n}$ and so $r_n \leq \rho(A)$. Since $r_{n-1} \leq r_n$ for all $n \in \mathbb{N}$ by (2.7) the proof is completed. ■

The following result from [5] generalizes and extends [41, Theorem 2.2 and Theorem 3.2(3)].

Proposition 2.1.4 *Let A be a positive kernel operator on $L^2(X, \mu)$ and $\alpha \in [0, 1]$. Then for all $r \in \{\rho, \rho_{ess}, \gamma, \|\cdot\|, w\}$ and $n \in \mathbb{N}$ we have*

$$r(S(A)) \leq r(S_\alpha(A)) \leq r(A) \text{ and} \quad (2.8)$$

$$r(S(A^n))^{\frac{1}{n}} \leq r(S_\alpha(A^n))^{\frac{1}{n}} \leq r(A). \quad (2.9)$$

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Proof: Since $S(K) = S(S_\alpha(K))$ Inequalities (2.8) follow from (2.3). Inequalities (2.9) follow from (2.8). ■

The following result which was established in [5] generalizes and extends [41, Theorems 2.3 and 3.3] and it's proved in a more general context in [7, Theorem 3]. The proof presented below uses that method.

Theorem 2.1.5 *Let A be a positive kernel operator on $L^2(X, \mu)$.*

For $r \in \{\rho, \rho_{ess}, \gamma, \|\cdot\|, w\}$ and $\alpha \in [0, 1]$ define $f_r(\alpha) = r(S_\alpha(A))$. Then f_r is decreasing in $[0, 0.5]$ and increasing in $[0.5, 1]$.

Proof: Assume $0 \leq \alpha_1 < \alpha_2 \leq \frac{1}{2}$ and let $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1}$. Then $\alpha \in (0, 1)$ and for every positive kernel operator A on $L^2(X, \mu)$ we have $S_{\alpha_2}(A) = S_\alpha(S_{\alpha_1}(A))$. Indeed, the kernel of the operator $S_\alpha(S_{\alpha_1}(A))$ is equal to

$$\begin{aligned} & (s_{\alpha_1}(a)(x, y))^\alpha (s_{\alpha_1}(a)(y, x))^{1-\alpha} \\ &= (a(x, y)^{\alpha_1} a(y, x)^{1-\alpha_1})^\alpha (a(y, x)^{\alpha_1} a(x, y)^{1-\alpha_1})^{1-\alpha} \\ &= a(x, y)^{\alpha_1 \alpha + (1-\alpha_1)(1-\alpha)} a(y, x)^{\alpha(1-\alpha_1) + \alpha_1(1-\alpha)} = a(x, y)^{\alpha_2} a(y, x)^{1-\alpha_2}, \end{aligned}$$

which is a kernel of the operator $S_{\alpha_2}(A)$ since

$$\alpha_1 \alpha + (1 - \alpha_1)(1 - \alpha) = \alpha(2\alpha_1 - 1) + 1 - \alpha_1 = \alpha_1 + \alpha_2 - 1 + 1 - \alpha_1 = \alpha_2$$

and

$$\alpha(1 - \alpha_1) + \alpha_1(1 - \alpha) = \alpha(1 - 2\alpha_1) + \alpha_1 = 1 - \alpha_1 - \alpha_2 + \alpha_1 = 1 - \alpha_2.$$

It follows from (2.3)

$$f_r(\alpha_2) = r(S_{\alpha_2}(A)) = r(S_\alpha(S_{\alpha_1}(A))) \leq r(S_{\alpha_1}(A)) = f_r(\alpha_1).$$

Similarly, in the case $\frac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1$ let $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_2 - 1}$. It follows that $\alpha \in (0, 1)$ and $S_{\alpha_1}(A) = S_\alpha(S_{\alpha_2}(A))$ for every positive kernel operator on $L^2(X, \mu)$. This holds since the kernel of $S_\alpha(S_{\alpha_2}(A))$ equals

$$\begin{aligned} & (s_{\alpha_2}(a)(x, y))^\alpha (s_{\alpha_2}(a)(y, x))^{1-\alpha} \\ &= (a(x, y)^{\alpha_2} a(y, x)^{1-\alpha_2})^\alpha (a(y, x)^{\alpha_2} a(x, y)^{1-\alpha_2})^{1-\alpha} \\ &= a(x, y)^{\alpha_2 \alpha + (1-\alpha_2)(1-\alpha)} a(y, x)^{\alpha(1-\alpha_2) + \alpha_2(1-\alpha)} = a(x, y)^{\alpha_1} a(y, x)^{1-\alpha_1}, \end{aligned}$$

which is a kernel of the operator $S_{\alpha_1}(A)$ since $\alpha_2\alpha + (1 - \alpha_2)(1 - \alpha) = \alpha_1$ and $\alpha(1 - \alpha_2) + \alpha_2(1 - \alpha) = 1 - \alpha_1$. From (2.3) we obtain in a similar way

$$f_r(\alpha_1) = r(S_{\alpha_1}(A)) = r(S_{\alpha}(S_{\alpha_2}(A))) \leq r(S_{\alpha_2}(A)) = f_r(\alpha_2)$$

which completes the proof. ■

2.2 Some refinements and generalizations of known inequalities

This section contains main results from [5].

The following refinement of inequalities (1.33) and (1.34) was proved in [38, Corollary 3.10].

Theorem 2.2.1 *Let $A_1, \dots, A_n$ be positive kernel operators on a Banach function space L . If $\alpha_1, \dots, \alpha_n$ are positive numbers such that $\sum_{i=1}^n \alpha_i = 1$ and if $m \in \mathbb{N}$ then*

$$r(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)}) \leq r((A_1^m)^{(\alpha_1)} \circ \dots \circ (A_n^m)^{(\alpha_n)})^{\frac{1}{m}} \leq r(A_1)^{\alpha_1} \dots r(A_n)^{\alpha_n} \quad (2.10)$$

for $r = \rho$

If, in addition, L and L^ have order continuous norms then Inequalities (2.10) hold also for $r = \rho_{ess}$.*

The following refinement is obtained by iterating (2.10).

Corollary 2.2.2 *Let $A_1, \dots, A_n$ be positive kernel operators on a Banach function space L . If $\alpha_1, \dots, \alpha_n$ are positive numbers such that $\sum_{i=1}^n \alpha_i = 1$ and if $m, l \in \mathbb{N}$ then*

$$\begin{aligned} r(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)}) &\leq r((A_1^m)^{(\alpha_1)} \circ \dots \circ (A_n^m)^{(\alpha_n)})^{\frac{1}{m}} \\ &\leq r((A_1^{ml})^{(\alpha_1)} \circ \dots \circ (A_n^{ml})^{(\alpha_n)})^{\frac{1}{ml}} \leq r(A_1)^{\alpha_1} \dots r(A_n)^{\alpha_n} \end{aligned} \quad (2.11)$$

for $r = \rho$.

If, in addition, L and L^ have order continuous norms then Inequalities (2.11) hold also for $r = \rho_{ess}$.*

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In case of sequence spaces $L \in \mathcal{L}$ it was proved in [5] (using standard methods from [10] and [34]) that inequalities (2.11) hold also under the condition $\sum_{i=1}^n \alpha_i \geq 1$. In this case additional refinements of (2.10) are also proved.

Theorem 2.2.3 *Given $L \in \mathcal{L}$, let $A_1, \dots, A_n$ be nonnegative matrices that define operators on L . If $\alpha_1, \dots, \alpha_n$ are nonnegative numbers such that $s_n = \sum_{i=1}^n \alpha_i \geq 1$ and if $m, l \in \mathbb{N}$ and $\beta_i = \frac{\alpha_i}{s_n}$ for all $i = 1, \dots, n$, then we have*

$$\begin{aligned} & \rho(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)}) \leq \rho((A_1^m)^{(\alpha_1)} \circ \dots \circ (A_n^m)^{(\alpha_n)})^{\frac{1}{m}} \\ & \leq \rho((A_1^{ml})^{(\alpha_1)} \circ \dots \circ (A_n^{ml})^{(\alpha_n)})^{\frac{1}{ml}} \leq \rho((A_1^{ml})^{(\beta_1)} \circ \dots \circ (A_n^{ml})^{(\beta_n)})^{\frac{s_n}{ml}} \\ & \leq \rho(A_1)^{\alpha_1} \dots \rho(A_n)^{\alpha_n} \end{aligned} \quad (2.12)$$

and

$$\begin{aligned} & \rho(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)}) \leq \rho((A_1^m)^{(\alpha_1)} \circ \dots \circ (A_n^m)^{(\alpha_n)})^{\frac{1}{m}} \\ & \leq \rho((A_1^m)^{(\beta_1)} \circ \dots \circ (A_n^m)^{(\beta_n)})^{\frac{s_n}{m}} \leq \rho((A_1^{ml})^{(\beta_1)} \circ \dots \circ (A_n^{ml})^{(\beta_n)})^{\frac{s_n}{ml}} \\ & \leq \rho(A_1)^{\alpha_1} \dots \rho(A_n)^{\alpha_n}. \end{aligned} \quad (2.13)$$

Proof: To prove that inequalities (2.11) hold under above assumptions we have by (1.27)

$$\begin{aligned} & \left(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)} \right)^m = \left(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)} \right) \dots \left(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)} \right) \\ & \leq (A_1^m)^{(\alpha_1)} \circ \dots \circ (A_n^m)^{(\alpha_n)}. \end{aligned} \quad (2.14)$$

It follows from (2.14) and (1.34) that

$$\begin{aligned} & \rho(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)})^m = \rho \left(\left(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)} \right)^m \right) \\ & \leq \rho((A_1^m)^{(\alpha_1)} \circ \dots \circ (A_n^m)^{(\alpha_n)}) \leq \rho(A_1^m)^{\alpha_1} \dots \rho(A_n^m)^{\alpha_n} = (\rho(A_1)^{\alpha_1} \dots \rho(A_n)^{\alpha_n})^m \end{aligned}$$

which proves (2.10) in this case. By iterating we obtain (2.11) under above assumptions.

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To prove (2.13), since $s_n \geq 1$ and $\alpha_i = \beta_i s_n$, it follows by the first inequality in (2.11) in case $s_n \geq 1$ using (1.39) and (1.34) that

$$\begin{aligned} & \rho(A_1^{(\alpha_1)} \circ \dots \circ A_n^{(\alpha_n)}) \leq \rho((A_1^m)^{(\alpha_1)} \circ \dots \circ (A_n^m)^{(\alpha_n)})^{\frac{1}{m}} \\ &= \rho\left(\left((A_1^m)^{(\beta_1)} \circ \dots \circ (A_n^m)^{(\beta_n)}\right)^{(s_n)}\right)^{\frac{1}{m}} \leq \rho((A_1^m)^{(\beta_1)} \circ \dots \circ (A_n^m)^{(\beta_n)})^{\frac{s_n}{m}} \\ &\leq \rho((A_1^{ml})^{(\beta_1)} \circ \dots \circ (A_n^{ml})^{(\beta_n)})^{\frac{s_n}{ml}} \leq (\rho(A_1^{ml})^{\alpha_1} \dots r(A_n^{ml})^{\alpha_n})^{\frac{1}{ml}} \\ &= \rho(A_1)^{\alpha_1} \dots r(A_n)^{\alpha_n}, \end{aligned}$$

which proves (2.13). Now (2.12) follows from (2.11) in case $s_n \geq 1$ and (2.13), which completes the proof. $\blacksquare$

The following result is a new variation of [37, Theorem 4.1] and was established in [5]. By σ_m is denoted the group of permutations of the set $\{1, \dots, m\}$.

Theorem 2.2.4 *Let m be even, $\{\tau, \nu\} \subset \sigma_m$ and let $H_1, \dots, H_m$ be positive kernel operators on $L^2(X, \mu)$. For $j = 1, \dots, \frac{m}{2}$ denote $A_j = H_{\tau(2j-1)}^* H_{\tau(2j)}$ and $A_{\frac{m}{2}+j} = A_j^* = H_{\tau(2j)}^* H_{\tau(2j-1)}$. Let $P_i = A_{\nu(i)} \dots A_{\nu(m)} A_{\nu(1)} \dots A_{\nu(i-1)}$ for $i = 1, \dots, m$.*

(i) *Then*

$$\begin{aligned} & \|H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})}\| \leq \rho(A_1^{(\frac{1}{m})} \circ \dots \circ A_m^{(\frac{1}{m})})^{\frac{1}{2}} \\ & \leq \rho\left(P_1^{(\frac{1}{m})} \circ P_2^{(\frac{1}{m})} \circ \dots \circ P_m^{(\frac{1}{m})}\right)^{\frac{1}{2m}} \leq \rho(A_{\nu(1)} \dots A_{\nu(m)})^{\frac{1}{2m}}. \end{aligned} \quad (2.15)$$

(ii) *If $H_1, \dots, H_m$ are nonnegative matrices that define operators on $l^2(R)$ and if $\alpha \geq \frac{1}{m}$, then*

$$\begin{aligned} & \|H_1^{(\alpha)} \circ \dots \circ H_m^{(\alpha)}\| \leq \rho(A_1^{(\alpha)} \circ \dots \circ A_m^{(\alpha)})^{\frac{1}{2}} \\ & \leq \rho\left(P_1^{(\alpha)} \circ P_2^{(\alpha)} \circ \dots \circ P_m^{(\alpha)}\right)^{\frac{1}{2m}} \leq \rho(A_{\nu(1)} \dots A_{\nu(m)})^{\frac{\alpha}{2}}. \end{aligned} \quad (2.16)$$

Proof: First to prove (2.15). By

$$\|H\| = \rho(H^* H)^{\frac{1}{2}} = \rho(H H^*)^{\frac{1}{2}}, \quad (2.17)$$

(1.29) and commutativity of Hadamard product we have

$$\begin{aligned}
 \|H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})}\| &= \rho((H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})})^* (H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})}))^{\frac{1}{2}} = \\
 &\rho[(((H_{\tau(1)}^*)^{(\frac{1}{m})} \circ \dots \circ (H_{\tau(m-1)}^*)^{(\frac{1}{m})} \circ (H_{\tau(2)}^*)^{(\frac{1}{m})} \circ \dots \circ (H_{\tau(m)}^*)^{(\frac{1}{m})})) \cdot \\
 &\quad (H_{\tau(2)}^{(\frac{1}{m})} \circ \dots \circ H_{\tau(m)}^{(\frac{1}{m})} \circ H_{\tau(1)}^{(\frac{1}{m})} \circ \dots \circ H_{\tau(m-1)}^{(\frac{1}{m})}))^{\frac{1}{2}} \\
 &\leq \rho((H_{\tau(1)}^* H_{\tau(2)})^{(\frac{1}{m})} \circ \dots \circ (H_{\tau(m-1)}^* H_{\tau(m)})^{(\frac{1}{m})} \circ (H_{\tau(2)}^* H_{\tau(1)})^{(\frac{1}{m})} \\
 &\quad \circ \dots \circ (H_{\tau(m)}^* H_{\tau(m-1)})^{(\frac{1}{m})})^{\frac{1}{2}} \\
 &= \rho(A_1^{(\frac{1}{m})} \circ \dots \circ A_m^{(\frac{1}{m})})^{\frac{1}{2}} = \rho(A_{\nu(1)}^{(\frac{1}{m})} \circ \dots \circ A_{\nu(m)}^{(\frac{1}{m})})^{\frac{1}{2}},
 \end{aligned}$$

which proves the first inequality in (2.2.4). The second and the third inequality in (2.15) follow from [37, Inequalities (4.2)]. Inequalities (2.16) are proved in a similar way by applying Theorem 1.2.16(ii). ■

By interchanging H_i with H_i^* for all i in Theorem 2.2.4 we obtain the following result, established in [5].

Corollary 2.2.5 *Let m be even, $\tau \in \sigma_m$, $\beta \in [0, 1]$ and let $H_1, \dots, H_m$ be positive kernel operators on $L^2(X, \mu)$. Let A_j for $j = 1, \dots, m$ be as in Theorem 2.2.4 and denote $B_j = H_{\tau(2j-1)} H_{\tau(2j)}^*$ and $B_{\frac{m}{2}+j} = B_j^* = H_{\tau(2j)} H_{\tau(2j-1)}^*$ for $j = 1, \dots, \frac{m}{2}$.*

(i) *Then*

$$\|H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})}\| \leq \rho(B_1^{(\frac{1}{m})} \circ \dots \circ B_m^{(\frac{1}{m})})^{\frac{1}{2}}$$

and

$$\|H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})}\| \leq \rho(A_1^{(\frac{1}{m})} \circ \dots \circ A_m^{(\frac{1}{m})})^{\frac{\beta}{2}} \rho(B_1^{(\frac{1}{m})} \circ \dots \circ B_m^{(\frac{1}{m})})^{\frac{1-\beta}{2}}.$$

(ii) *If $H_1, \dots, H_m$ are nonnegative matrices that define operators on $l^2(R)$ and if $\alpha \geq \frac{1}{m}$, then*

$$\|H_1^{(\alpha)} \circ \dots \circ H_m^{(\alpha)}\| \leq \rho(B_1^{(\alpha)} \circ \dots \circ B_m^{(\alpha)})^{\frac{1}{2}}$$

and

$$\|H_1^{(\alpha)} \circ \dots \circ H_m^{(\alpha)}\| \leq \rho(A_1^{(\alpha)} \circ \dots \circ A_m^{(\alpha)})^{\frac{\beta}{2}} \rho(B_1^{(\alpha)} \circ \dots \circ B_m^{(\alpha)})^{\frac{1-\beta}{2}}.$$

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The following two results extend, generalize and refine [46, Theorem 2.8] and give an extension and a different refinement of [37, Inequality(4.16)] in the case $\alpha \geq \frac{2}{m}$ and were also established in [5].

Theorem 2.2.6 *Let m be even, $\alpha \geq \frac{2}{m}$, $\tau \in \sigma_m$ and let $H_1, \dots, H_m$ be non-negative matrices that define operators on $l^2(R)$. Let A_j for $j = 1, \dots, m$ be as in Theorem 2.2.4 and denote $S_i = A_i \cdots A_{\frac{m}{2}} A_1 \cdots A_{i-1}$ for $i = 1, \dots, \frac{m}{2}$. Then*

$$\begin{aligned} & \|H_1^{(\alpha)} \circ \cdots \circ H_m^{(\alpha)}\| \leq \rho(A_1^{(\alpha)} \circ \cdots \circ A_m^{(\alpha)})^{\frac{1}{2}} \leq \rho(A_1^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)}) \\ & = \rho((H_{\tau(1)}^* H_{\tau(2)})^{(\alpha)} \circ (H_{\tau(3)}^* H_{\tau(4)})^{(\alpha)} \circ \cdots \circ (H_{\tau(m-1)}^* H_{\tau(m)})^{(\alpha)}) \\ & \leq \rho\left(S_1^{(\alpha)} \circ S_2^{(\alpha)} \circ \cdots \circ S_{\frac{m}{2}}^{(\alpha)}\right)^{\frac{2}{m}} \leq \rho(H_{\tau(1)}^* H_{\tau(2)} H_{\tau(3)}^* H_{\tau(4)} \cdots H_{\tau(m-1)}^* H_{\tau(m)})^{\alpha}. \end{aligned} \quad (2.18)$$

Proof: By the first inequality in (2.16) and Theorem 1.2.16(ii) we have

$$\begin{aligned} & \|H_1^{(\alpha)} \circ \cdots \circ H_m^{(\alpha)}\| \leq \rho(A_1^{(\alpha)} \circ \cdots \circ A_m^{(\alpha)})^{\frac{1}{2}} \\ & = \rho(A_1^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)} \circ (A_1^*)^{(\alpha)} \circ \cdots \circ (A_{\frac{m}{2}}^*)^{(\alpha)})^{\frac{1}{2}} \\ & \leq (\rho(A_1^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)}) \rho((A_1^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)})^*))^{\frac{1}{2}} = \rho(A_1^{(\alpha)} \circ \cdots \circ A_{\frac{m}{2}}^{(\alpha)}) \\ & = \rho((H_{\tau(1)}^* H_{\tau(2)})^{(\alpha)} \circ (H_{\tau(3)}^* H_{\tau(4)})^{(\alpha)} \circ \cdots \circ (H_{\tau(m-1)}^* H_{\tau(m)})^{(\alpha)}). \end{aligned}$$

Since

$$\begin{aligned} & ((H_{\tau(1)}^* H_{\tau(2)})^{(\alpha)} \circ (H_{\tau(3)}^* H_{\tau(4)})^{(\alpha)} \circ \cdots \circ (H_{\tau(m-1)}^* H_{\tau(m)})^{(\alpha)})^{\frac{m}{2}} = \\ & ((H_{\tau(1)}^* H_{\tau(2)})^{(\alpha)} \circ \cdots \circ (H_{\tau(m-1)}^* H_{\tau(m)})^{(\alpha)}) ((H_{\tau(3)}^* H_{\tau(4)})^{(\alpha)} \circ \cdots \circ (H_{\tau(1)}^* H_{\tau(2)})^{(\alpha)}) \\ & \quad \cdots ((H_{\tau(m-1)}^* H_{\tau(m)})^{(\alpha)} \circ \cdots \circ (H_{\tau(m-3)}^* H_{\tau(m-2)})^{(\alpha)}), \end{aligned}$$

we obtain by (1.29) that

$$\begin{aligned} & \rho((H_{\tau(1)}^* H_{\tau(2)})^{(\alpha)} \circ (H_{\tau(3)}^* H_{\tau(4)})^{(\alpha)} \circ \cdots \circ (H_{\tau(m-1)}^* H_{\tau(m)})^{(\alpha)}) \leq \\ & \rho(S_1^{(\alpha)} \circ S_2^{(\alpha)} \circ \cdots \circ S_{\frac{m}{2}}^{(\alpha)})^{\frac{2}{m}} \leq (\rho(S_1)^{\alpha} \cdots \rho(S_{\frac{m}{2}})^{\alpha})^{\frac{2}{m}} \\ & = \rho(H_{\tau(1)}^* H_{\tau(2)} H_{\tau(3)}^* H_{\tau(4)} \cdots H_{\tau(m-1)}^* H_{\tau(m)})^{\alpha}, \end{aligned}$$

where the last equality follows from $\rho(S_1) = \cdots = \rho(S_{\frac{m}{2}})$. ■

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Corollary 2.2.7 *Let m be even, $\alpha \geq \frac{2}{m}$, $\tau \in \sigma_m$, $\beta \in [0, 1]$ and let $H_1, \dots, H_m$ be nonnegative matrices that define operators on $l^2(R)$. Let A_j and B_j for $j = 1, \dots, m$ be as in Corollary 2.2.5 and denote $S_i = A_i \cdots A_{\frac{m}{2}} A_1 \cdots A_{i-1}$ and $T_i = B_i \cdots B_{\frac{m}{2}} B_1 \cdots B_{i-1}$ for $i = 1, \dots, \frac{m}{2}$. Then*

$$\begin{aligned} \|H_1^{(\alpha)} \circ \cdots \circ H_m^{(\alpha)}\| &\leq \rho(A_1^{(\alpha)} \circ \cdots \circ A_m^{(\alpha)})^{\frac{\beta}{2}} \rho(B_1^{(\alpha)} \circ \cdots \circ B_m^{(\alpha)})^{\frac{1-\beta}{2}} \quad (2.19) \\ &\leq \rho((H_{\tau(1)}^* H_{\tau(2)})^{(\alpha)} \circ (H_{\tau(3)}^* H_{\tau(4)})^{(\alpha)} \circ \cdots \circ (H_{\tau(m-1)}^* H_{\tau(m)})^{(\alpha)})^{\beta} \cdot \\ &\quad \rho((H_{\tau(1)} H_{\tau(2)}^*)^{(\alpha)} \circ (H_{\tau(3)} H_{\tau(4)}^*)^{(\alpha)} \circ \cdots \circ (H_{\tau(m-1)} H_{\tau(m)}^*)^{(\alpha)})^{1-\beta} \\ &\leq \rho\left(S_1^{(\alpha)} \circ S_2^{(\alpha)} \circ \cdots \circ S_{\frac{m}{2}}^{(\alpha)}\right)^{\frac{2\beta}{m}} \rho\left(T_1^{(\alpha)} \circ T_2^{(\alpha)} \circ \cdots \circ T_{\frac{m}{2}}^{(\alpha)}\right)^{\frac{2(1-\beta)}{m}} \\ &\leq \rho(H_{\tau(1)}^* H_{\tau(2)} H_{\tau(3)}^* H_{\tau(4)} \cdots H_{\tau(m-1)}^* H_{\tau(m)})^{\alpha\beta} \rho(H_{\tau(1)} H_{\tau(2)}^* H_{\tau(3)} H_{\tau(4)}^* \\ &\quad \cdots H_{\tau(m-1)} H_{\tau(m)}^*)^{\alpha(1-\beta)}. \end{aligned}$$

The following result, established in [5], extends [46, Theorem 2.13] and [37, Theorem 4.1].

Theorem 2.2.8 *Let $H_1, \dots, H_m$ be positive kernel operators on $L^2(X, \mu)$ and $\{\tau, \nu\} \subset \sigma_m$.*

Denote $Q_j = H_{\tau(j)}^ H_{\nu(j)} \cdots H_{\tau(m)}^* H_{\nu(m)} H_{\tau(1)}^* H_{\nu(1)} \cdots H_{\tau(j-1)}^* H_{\nu(j-1)}$ for $j = 1, \dots, m$.*

(i) Then

$$\begin{aligned} \|H_1^{(\frac{1}{m})} \circ \cdots \circ H_m^{(\frac{1}{m})}\| &\leq \rho((H_{\tau(1)}^* H_{\nu(1)})^{(\frac{1}{m})} \circ \cdots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})})^{\frac{1}{2}} \\ &\leq \rho(Q_1^{(\frac{1}{m})} \circ \cdots \circ Q_m^{(\frac{1}{m})})^{\frac{1}{2m}} \leq \rho(H_{\tau(1)}^* H_{\nu(1)} \cdots H_{\tau(m)}^* H_{\nu(m)})^{\frac{1}{2m}}. \quad (2.20) \end{aligned}$$

(ii) If $H_1, \dots, H_m$ are nonnegative matrices that define operators on $l^2(R)$ and if $\alpha \geq \frac{1}{m}$, then

$$\begin{aligned} \|H_1^{(\alpha)} \circ \cdots \circ H_m^{(\alpha)}\| &\leq \rho((H_{\tau(1)}^* H_{\nu(1)})^{(\alpha)} \circ \cdots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\alpha)})^{\frac{1}{2}} \\ &\leq \rho(Q_1^{(\alpha)} \circ \cdots \circ Q_m^{(\alpha)})^{\frac{1}{2m}} \leq \rho(H_{\tau(1)}^* H_{\nu(1)} \cdots H_{\tau(m)}^* H_{\nu(m)})^{\frac{\alpha}{2}}. \quad (2.21) \end{aligned}$$

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Proof: Let us first prove (2.20). By (2.17) and (1.29) it follows

$$\begin{aligned} \|H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})}\| &= \rho((H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})})^*(H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})}))^{\frac{1}{2}} = \\ &= \rho(((H_{\tau(1)}^*)^{(\frac{1}{m})} \circ \dots \circ (H_{\tau(m)}^*)^{(\frac{1}{m})})((H_{\nu(1)})^{(\frac{1}{m})} \circ \dots \circ (H_{\nu(m)})^{(\frac{1}{m})}))^{\frac{1}{2}} \\ &\leq \rho((H_{\tau(1)}^* H_{\nu(1)})^{(\frac{1}{m})} \circ \dots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})})^{\frac{1}{2}}. \end{aligned}$$

Notice that

$$\begin{aligned} ((H_{\tau(1)}^* H_{\nu(1)})^{(\frac{1}{m})} \circ \dots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})})^m &= ((H_{\tau(1)}^* H_{\nu(1)})^{(\frac{1}{m})} \\ &\quad \circ \dots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})}) \\ &\quad ((H_{\tau(2)}^* H_{\nu(2)})^{(\frac{1}{m})} \circ \dots \circ (H_{\tau(1)}^* H_{\nu(1)})^{(\frac{1}{m})}) \\ &\quad \dots ((H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})} \circ \dots \circ (H_{\tau(m-1)}^* H_{\nu(m-1)})^{(\frac{1}{m})}). \end{aligned}$$

It follows by (1.29) that

$$\begin{aligned} \rho((H_{\tau(1)}^* H_{\nu(1)})^{(\frac{1}{m})} \circ \dots \circ (H_{\tau(m)}^* H_{\nu(m)})^{(\frac{1}{m})})^{\frac{1}{2}} &\leq \rho(Q_1^{(\frac{1}{m})} \circ \dots \circ Q_m^{(\frac{1}{m})})^{\frac{1}{2m}} \\ &\leq (\rho(Q_1) \dots \rho(Q_m))^{\frac{1}{2m^2}} = \rho(H_{\tau(1)}^* H_{\nu(1)} \dots H_{\tau(m)}^* H_{\nu(m)})^{\frac{1}{2m}}, \end{aligned}$$

where the last equality follows from $\rho(Q_1) = \dots = \rho(Q_m)$. This completes the proof of (2.20). The proof of (2.21) is similar by applying Theorem 1.2.16(ii). ■

The following corollary is a refinement of [37, Inequality (4.11)], which differs from refinements in [37, Inequalities (4.15) and (4.17)]. It also extends and generalizes [46, Corollary 2.15] and was obtained in [5].

Corollary 2.2.9 *Let m be odd and let $H_1, \dots, H_m$ be positive kernel operators on $L^2(X, \mu)$.*

(i) *Then*

$$\begin{aligned} &\|H_1^{(\frac{1}{m})} \circ \dots \circ H_m^{(\frac{1}{m})}\| \\ &\leq \rho((H_1^* H_2)^{(\frac{1}{m})} \circ \dots \circ (H_{m-2}^* H_{m-1})^{(\frac{1}{m})} \circ (H_m^* H_1)^{(\frac{1}{m})} \circ (H_2^* H_3)^{(\frac{1}{m})} \circ \\ &\quad \dots \circ (H_{m-1}^* H_m)^{(\frac{1}{m})})^{\frac{1}{2}} \\ &\leq \rho(H_1^* H_2 \dots H_{m-2}^* H_{m-1} H_m^* H_1 H_2^* H_3 \dots H_{m-1}^* H_m)^{\frac{1}{2m}}. \end{aligned} \tag{2.22}$$

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(ii) If $H_1, \dots, H_m$ are nonnegative matrices that define operators on $l^2(R)$ and if $\alpha \geq \frac{1}{m}$, then

$$\begin{aligned} & \|H_1^{(\alpha)} \circ \dots \circ H_m^{(\alpha)}\| \\ & \leq \rho((H_1^* H_2)^{(\alpha)} \circ \dots \circ (H_{m-2}^* H_{m-1})^{(\alpha)} \circ (H_m^* H_1)^{(\alpha)} \circ (H_2^* H_3)^{(\alpha)} \circ \\ & \quad \dots \circ (H_{m-1}^* H_m)^{(\alpha)})^{\frac{1}{2}} \\ & \leq \rho(H_1^* H_2 \cdots H_{m-2}^* H_{m-1} H_m^* H_1 H_2^* H_3 \cdots H_{m-1}^* H_m)^{\frac{\alpha}{2}}. \end{aligned} \quad (2.23)$$

Proof: The result follows by taking the permutations $\tau(j) = 2j - 1$ for $1 \leq j \leq \frac{m+1}{2}$; $\tau(j) = 2(j - \frac{m+1}{2})$ for $\frac{m+3}{2} \leq j \leq m$ and $\nu(j) = 2j$ for $1 \leq j \leq \frac{m-1}{2}$; $\nu(j) = 2(j - \frac{m-1}{2}) - 1$ for $\frac{m+1}{2} \leq j \leq m$ in Theorem 2.2.8. ■

The following corollary, obtained in [5], gives new lower bounds for the operator norm of the Jordan triple product ABA which differ from the one obtained in [37, Corollary 4.10]. The result follows from Corollary 2.2.9 and Theorem 2.2.8 by taking $H_1 = A$, $H_2 = B^*$ and $H_3 = A$.

Corollary 2.2.10 *Let A and B be positive kernel operators on $L^2(X, \mu)$.*

(i) *Then*

$$\begin{aligned} & \|A^{(\frac{1}{3})} \circ (B^*)^{(\frac{1}{3})} \circ A^{(\frac{1}{3})}\| \\ & \leq \rho^{\frac{1}{2}} \left((A^* B^*)^{(\frac{1}{3})} \circ (A^* A)^{(\frac{1}{3})} \circ (BA)^{(\frac{1}{3})} \right) \\ & \leq \rho^{\frac{1}{6}} \left((A^* B^* A^* ABA)^{(\frac{1}{3})} \circ (A^* ABAA^* B^*)^{(\frac{1}{3})} \circ (BAA^* B^* A^* A)^{(\frac{1}{3})} \right) \\ & \leq \|ABA\|^{\frac{1}{3}}. \end{aligned} \quad (2.24)$$

(ii) *If A and B are nonnegative matrices that define operators on $l^2(R)$ and if $\alpha \geq \frac{1}{3}$, then*

$$\begin{aligned} & \|A^{(\alpha)} \circ (B^*)^{(\alpha)} \circ A^{(\alpha)}\| \\ & \leq \rho^{\frac{1}{2}} \left((A^* B^*)^{(\alpha)} \circ (A^* A)^{(\alpha)} \circ (BA)^{(\alpha)} \right) \\ & \leq \rho^{\frac{1}{6}} \left((A^* B^* A^* ABA)^{(\alpha)} \circ (A^* ABAA^* B^*)^{(\alpha)} \circ (BAA^* B^* A^* A)^{(\alpha)} \right) \\ & \leq \|ABA\|^{\alpha}. \end{aligned} \quad (2.25)$$

The following result, established in [5], generalizes [46, Inequality (2.12)].

Lemma 2.2.11 *Let $\alpha \geq \frac{1}{2}$ and let C be a nonnegative matrix that defines an operator on $l^2(R)$. Then*

$$\rho(C^{(\alpha)} \circ (C^*)^{(\alpha)}) \leq \rho(C^{(\alpha)} \circ C^{(\alpha)}) \leq \rho(C)^{2\alpha}.$$

Proof: By applying Theorem 1.2.16(ii) twice it follows

$$\begin{aligned} \rho(C^{(\alpha)} \circ (C^*)^{(\alpha)}) &= \rho((C^{(\alpha)} \circ C^{(\alpha)})^{(\frac{1}{2})} \circ ((C^*)^{(\alpha)} \circ (C^*)^{(\alpha)})^{(\frac{1}{2})}) \\ &\leq \rho(C^{(\alpha)} \circ C^{(\alpha)})^{\frac{1}{2}} \rho((C^*)^{(\alpha)} \circ (C^*)^{(\alpha)})^{\frac{1}{2}} = \rho(C^{(\alpha)} \circ C^{(\alpha)}) \leq \rho(C)^{2\alpha}, \end{aligned}$$

which completes the proof. ■

The following result, obtained in [5], generalizes [46, Theorem 2.17] and refines [37, Inequalities (4.9)]. It follows from Theorem 2.2.8 (or [37, Inequalities (4.9)]) and Lemma 2.2.11.

Corollary 2.2.12 *Let $\alpha \geq \frac{1}{2}$ and let A and B be nonnegative matrices that define operators on $l^2(R)$. Then*

$$\begin{aligned} \|A^{(\alpha)} \circ B^{(\alpha)}\| &\leq \rho^{\frac{1}{2}}((A^*B)^{(\alpha)} \circ (B^*A)^{(\alpha)}) \\ &\leq \rho^{\frac{1}{2}}((A^*B)^{(\alpha)} \circ (A^*B)^{(\alpha)}) \leq \rho^\alpha(A^*B). \end{aligned} \tag{2.26}$$

2.3 Results on the joint and generalized spectral radius and their essential versions

In this section we present some results obtained in [6].

In [38] and later it remained unnoticed that several inequalities in Theorem 1.4.2 are in fact equalities and this result was established in [6].

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Theorem 2.3.1 *Let Ψ and Σ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \dots, \alpha_m$ be positive numbers such that $\sum_{j=1}^m \alpha_j = 1$.*

(i) *If $r \in \{\rho, \hat{\rho}\}$ and $\beta \in [0, 1]$, then*

$$r(\Psi) = r(\Psi^{(\alpha_1)} \circ \dots \circ \Psi^{(\alpha_m)}) \quad (2.27)$$

and

$$\begin{aligned} r(\Psi\Sigma) &= r((\Psi^{(\frac{1}{2})} \circ \Psi^{(\frac{1}{2})})(\Sigma^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})})) \\ &= r\left((\Psi\Sigma)^{(\frac{1}{2})} \circ (\Psi\Sigma)^{(\frac{1}{2})}\right)^\beta r\left((\Sigma\Psi)^{(\frac{1}{2})} \circ (\Sigma\Psi)^{(\frac{1}{2})}\right)^{1-\beta}. \end{aligned} \quad (2.28)$$

If, in addition, L and L^* have order continuous norms, then (2.27) and (2.28) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

(ii) *If $L \in \mathcal{L}$, $r \in \{\rho, \hat{\rho}\}$, $m, n \in \mathbb{N}$, $\alpha \geq 1$ and if Ψ is a bounded set of nonnegative matrices that define operators on L , then*

$$r(\Psi^{(m)}) \leq r(\Psi \circ \dots \circ \Psi) \leq r(\Psi^n \circ \dots \circ \Psi^n)^{\frac{1}{n}} \leq r(\Psi)^m, \quad (2.29)$$

where in (2.29) the Hadamard products in $\Psi \circ \dots \circ \Psi$ and in $\Psi^n \circ \dots \circ \Psi^n$ are taken m times, and

$$r(\Psi^{(\alpha)}) \leq r(\Psi^{(\alpha-1)} \circ \Psi) \leq r((\Psi^n)^{(\alpha-1)} \circ \Psi^n)^{\frac{1}{n}} \leq r(\Psi)^\alpha. \quad (2.30)$$

Proof: (i) To prove (2.27) first observe that $\Psi \subset \Psi^{(\alpha_1)} \circ \dots \circ \Psi^{(\alpha_m)}$, since $A = A^{(\alpha_1)} \circ \dots \circ A^{(\alpha_m)}$ for all $A \in \Psi$. It follows that

$$r(\Psi) \leq r(\Psi^{(\alpha_1)} \circ \dots \circ \Psi^{(\alpha_m)}) \leq r(\Psi)^{\alpha_1} \dots r(\Psi)^{\alpha_m} = r(\Psi)$$

by Theorem 1.4.1 and so $r(\Psi) = r(\Psi^{(\alpha_1)} \circ \dots \circ \Psi^{(\alpha_m)})$.

Similary, to prove (2.28) observe that $\Psi\Sigma \subset (\Psi^{(\frac{1}{2})} \circ \Psi^{(\frac{1}{2})})(\Sigma^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})})$, since $AB = (A^{(\frac{1}{2})} \circ A^{(\frac{1}{2})})(B^{(\frac{1}{2})} \circ B^{(\frac{1}{2})})$ for all $A \in \Psi$ and $B \in \Sigma$. It follows that

$$\begin{aligned} r(\Psi\Sigma) &\leq r((\Psi^{(\frac{1}{2})} \circ \Psi^{(\frac{1}{2})})(\Sigma^{(\frac{1}{2})} \circ \Sigma^{(\frac{1}{2})})) \\ &\leq r\left((\Psi\Sigma)^{(\frac{1}{2})} \circ (\Psi\Sigma)^{(\frac{1}{2})}\right)^\beta r\left((\Sigma\Psi)^{(\frac{1}{2})} \circ (\Sigma\Psi)^{(\frac{1}{2})}\right)^{1-\beta} \leq r(\Psi\Sigma) \end{aligned}$$

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by (1.66), which proves (2.28). It is proved similarly that (2.27) and (2.28) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ in the case when L and L^* have order continuous norms.

(ii) For the proof of (2.29) observe that $\Psi^{(m)} \subset \Psi \circ \dots \circ \Psi$, since $A^{(m)} = A \circ \dots \circ A$ for all $A \in \Psi$. By Theorem 1.4.3, Inequalities (2.29) follow. Inequalities (2.30) are proved in a similar way.

The following result was established in [6] and extends Inequalities (1.29) and (1.63) and Theorem 1.4.3. ■

Theorem 2.3.2 *Let $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \dots, \alpha_m$ be positive numbers.*

(i) *If $r \in \{\rho, \hat{\rho}\}$, $\sum_{i=1}^m \alpha_i = 1$ and $n \in \mathbb{N}$, then*

$$\begin{aligned} & r \left(\left(\Psi_{11}^{(\alpha_1)} \circ \dots \circ \Psi_{1m}^{(\alpha_m)} \right) \dots \left(\Psi_{k1}^{(\alpha_1)} \circ \dots \circ \Psi_{km}^{(\alpha_m)} \right) \right) \\ & \leq r \left((\Psi_{11} \dots \Psi_{k1})^{(\alpha_1)} \circ \dots \circ (\Psi_{1m} \dots \Psi_{km})^{(\alpha_m)} \right) \\ & \leq r \left(((\Psi_{11} \dots \Psi_{k1})^n)^{(\alpha_1)} \circ \dots \circ ((\Psi_{1m} \dots \Psi_{km})^n)^{(\alpha_m)} \right)^{\frac{1}{n}} \\ & \leq r(\Psi_{11} \dots \Psi_{k1})^{\alpha_1} \dots r(\Psi_{1m} \dots \Psi_{km})^{\alpha_m}. \end{aligned} \quad (2.31)$$

If, in addition, L and L^ have order continuous norms, then Inequalities (2.31) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.*

(ii) *If $L \in \mathcal{L}$, $r \in \{\rho, \hat{\rho}\}$, $\sum_{j=1}^m \alpha_j \geq 1$ and $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ are bounded sets of nonnegative matrices that define positive operators on L , then Inequalities (2.31) hold.*

In particular, if $\Psi_1, \dots, \Psi_k$ are bounded sets of nonnegative matrices that define positive operators on L and $t \geq 1$, then

$$r(\Psi_1^{(t)} \dots \Psi_k^{(t)}) \leq r((\Psi_1 \dots \Psi_k)^{(t)}) \leq r(((\Psi_1 \dots \Psi_k)^n)^{(t)})^{\frac{1}{n}} \leq r(\Psi_1 \dots \Psi_k)^t. \quad (2.32)$$

Proof: (i) Let $r \in \{\rho, \hat{\rho}\}$, $\sum_{i=1}^m \alpha_i = 1$ and $n \in \mathbb{N}$. To prove the first inequality in (2.31) let $l \in \mathbb{N}$ and

$$A \in \left(\left(\Psi_{11}^{(\alpha_1)} \circ \dots \circ \Psi_{1m}^{(\alpha_m)} \right) \dots \left(\Psi_{k1}^{(\alpha_1)} \circ \dots \circ \Psi_{km}^{(\alpha_m)} \right) \right)^l.$$

Then $A = A_1 \dots A_l$, where for each $i = 1, \dots, l$ we have

$$A_i = \left(A_{i11}^{(\alpha_1)} \circ \dots \circ A_{i1m}^{(\alpha_m)} \right) \dots \left(A_{ik1}^{(\alpha_1)} \circ \dots \circ A_{ikm}^{(\alpha_m)} \right),$$

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where $A_{i11} \in \Psi_{11}, \dots, A_{i1m} \in \Psi_{1m}, \dots, A_{ik1} \in \Psi_{k1}, \dots, A_{ikm} \in \Psi_{km}$. Then by (1.27) for each $i = 1, \dots, l$ we have

$$A_i \leq C_i := (A_{i11}A_{i21} \cdots A_{ik1})^{(\alpha_1)} \circ \cdots \circ (A_{i1m}A_{i2m} \cdots A_{ikm})^{(\alpha_m)},$$

where $C_i \in (\Psi_{11} \cdots \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} \cdots \Psi_{km})^{(\alpha_m)}$. Therefore

$$A \leq C := C_1 \cdots C_l \in ((\Psi_{11} \cdots \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} \cdots \Psi_{km})^{(\alpha_m)})^l,$$

$\rho(A)^{1/l} \leq \rho(C)^{1/l}$ and $\|A\|^{1/l} \leq \|C\|^{1/l}$, which implies the first inequality in (2.31). The second and third inequality in (2.31) follow from (1.63).

If, in addition, L and L^* have order continuous norms and $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$, then Inequalities (2.31) are proved similarly. Under the assumptions of (ii) Inequalities (2.31) are proved in a similar way by applying Theorems 1.2.16(ii) and 1.4.3. $\blacksquare$

Next it follows an extension of Theorem 1.4.2 by refining (1.64), obtained in [6].

Theorem 2.3.3 *Let $\Psi_1, \dots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\Phi_j = \Psi_j \cdots \Psi_m \Psi_1 \cdots \Psi_{j-1}$ for $j = 1, \dots, m$. If $r \in \{\rho, \hat{\rho}\}$, then*

$$\begin{aligned} r \left(\Psi_1^{(\frac{1}{m})} \circ \Psi_2^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})} \right) &\leq r \left(\Phi_1^{(\frac{1}{m})} \circ \Phi_2^{(\frac{1}{m})} \circ \cdots \circ \Phi_m^{(\frac{1}{m})} \right)^{\frac{1}{m}} \\ &\leq r \left((\Phi_1^n)^{(\frac{1}{m})} \circ (\Phi_2^n)^{(\frac{1}{m})} \circ \cdots \circ (\Phi_m^n)^{(\frac{1}{m})} \right)^{\frac{1}{nm}} \leq r(\Psi_1 \Psi_2 \cdots \Psi_m)^{\frac{1}{m}}. \end{aligned} \quad (2.33)$$

If, in addition, L and L^* have order continuous norms, then Inequalities (2.33) are valid also for all $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

Proof: Let $r \in \{\rho, \hat{\rho}\}$. Denote

$$\begin{aligned} \Sigma_1 &= \Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}, \quad \Sigma_2 = \Psi_2^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})} \circ \Psi_1^{(\frac{1}{m})}, \dots, \\ \Sigma_m &= \Psi_m^{(\frac{1}{m})} \circ \Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_{m-1}^{(\frac{1}{m})}. \end{aligned}$$

Then by (1.62), (2.31) and commutativity of Hadamard product we have

$$\begin{aligned} r \left(\Psi_1^{(\frac{1}{m})} \circ \Psi_2^{(\frac{1}{m})} \circ \dots \circ \Psi_m^{(\frac{1}{m})} \right)^m &= r \left(\left(\Psi_1^{(\frac{1}{m})} \circ \Psi_2^{(\frac{1}{m})} \circ \dots \circ \Psi_m^{(\frac{1}{m})} \right)^m \right) \\ &= r(\Sigma_1 \Sigma_2 \dots \Sigma_m) \leq r \left(\Phi_1^{(\frac{1}{m})} \circ \Phi_2^{(\frac{1}{m})} \circ \dots \circ \Phi_m^{(\frac{1}{m})} \right), \end{aligned}$$

which proves the first inequality in (2.33). The second and the third inequality in (2.33) follow from (1.63) (or from (2.31)), since $r(\Phi_1) = r(\Phi_2) = \dots r(\Phi_m) = r(\Psi_1 \Psi_2 \dots \Psi_m)$ by (1.62). If, in addition, L and L^* have order continuous norms, then (2.33) for $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$ is proved in a similar way. $\blacksquare$

The following result extends (2.28) and was obtained in [6].

Theorem 2.3.4 *Let $\Psi_1, \dots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \dots, \alpha_m$ be nonnegative numbers such that $\sum_{j=1}^m \alpha_j = 1$. If $\Phi_j = \Psi_j \dots \Psi_m \Psi_1 \dots \Psi_{j-1}$ for $j = 1, \dots, m$, $\beta \in [0, 1]$, then for all $r \in \{\rho, \hat{\rho}\}$ we have*

$$\begin{aligned} r(\Psi_1 \Psi_2 \dots \Psi_m) &= r \left(\left(\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)} \right) \dots \left(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)} \right) \right) \\ &= r \left(\Phi_1^{(\beta)} \circ \Phi_1^{(1-\beta)} \right)^{\alpha_1} \dots r \left(\Phi_m^{(\beta)} \circ \Phi_m^{(1-\beta)} \right)^{\alpha_m}. \end{aligned} \quad (2.34)$$

If, in addition, L and L^* have order continuous norms, then Equalities (2.34) are valid for $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

Proof: Let $r \in \{\rho, \hat{\rho}\}$. To prove Equalities (2.34) we use the first inequality in (2.31) and (1.62) to obtain that

$$r \left(\left(\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)} \right) \dots \left(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)} \right) \right) \leq r \left(\Phi_i^{(\beta)} \circ \Phi_i^{(1-\beta)} \right) \quad (2.35)$$

for all $i = 1, \dots, m$. Indeed, by (1.62) and the first inequality in (2.31) we have

$$\begin{aligned} &r \left(\left(\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)} \right) \dots \left(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)} \right) \right) \\ &= r \left(\left(\Psi_i^{(\beta)} \circ \Psi_i^{(1-\beta)} \right) \dots \left(\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)} \right) \left(\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)} \right) \dots \left(\Psi_{i-1}^{(\beta)} \circ \Psi_{i-1}^{(1-\beta)} \right) \right) \\ &\leq r \left(\Phi_i^{(\beta)} \circ \Phi_i^{(1-\beta)} \right), \end{aligned}$$

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which proves (2.35). Since $\sum_{j=1}^m \alpha_j = 1$, Inequality (2.35) implies

$$\begin{aligned} r((\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)}) \cdots (\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)})) &\leq r(\Phi_1^{(\beta)} \circ \Phi_1^{(1-\beta)})^{\alpha_1} \cdots r(\Phi_m^{(\beta)} \circ \Phi_m^{(1-\beta)})^{\alpha_m} \\ &\leq r(\Psi_1 \cdots \Psi_m). \end{aligned} \quad (2.36)$$

The second inequality in (2.36) follows from (1.63) and the fact that $r(\Phi_1) = \cdots = r(\Phi_m) = r(\Psi_1 \cdots \Psi_m)$. Since $\Psi_i \subset \Psi_i^{(\beta)} \circ \Psi_i^{(1-\beta)}$ for all $i = 1, \dots, m$ and $\beta \in [0, 1]$, we obtain

$$r(\Psi_1 \cdots \Psi_m) \leq r((\Psi_1^{(\beta)} \circ \Psi_1^{(1-\beta)}) \cdots (\Psi_m^{(\beta)} \circ \Psi_m^{(1-\beta)})),$$

which together with (2.36) proves Equalities (2.34). If, in addition, L and L^* have order continuous norms, then Equalities (2.34) are proved in a similar way for $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$. ■

The following result, established in [6] that extends main result from [35], is proved in a similar way as Theorem 2.3.3 by applying Theorems 1.4.3 and 2.3.2(ii) instead of Theorems 1.4.1 and 2.3.2(i) in the proofs above.

Theorem 2.3.5 *Given $L \in \mathcal{L}$, let $\Psi_1, \dots, \Psi_m$ be bounded sets of nonnegative matrices that define operators on L and $\Phi_j = \Psi_j \cdots \Psi_m \Psi_1 \cdots \Psi_{j-1}$ for $j = 1, \dots, m$. Assume that $\alpha \geq \frac{1}{m}$, $\alpha_j \geq 0$, $j = 1, \dots, m$, $\sum_{j=1}^m \alpha_j \geq 1$ and $n \in \mathbb{N}$. If $r \in \{\rho, \hat{\rho}\}$ and $\Sigma_j = \Psi_j^{(\alpha m)} \cdots \Psi_m^{(\alpha m)} \Psi_1^{(\alpha m)} \cdots \Psi_{j-1}^{(\alpha m)}$ for $j = 1, \dots, m$, then we have*

$$\begin{aligned} r(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}) &\leq r(\Phi_1^{(\alpha)} \circ \cdots \circ \Phi_m^{(\alpha)})^{\frac{1}{m}} \\ &\leq r((\Phi_1^n)^{(\alpha)} \circ \cdots \circ (\Phi_m^n)^{(\alpha)})^{\frac{1}{mn}} \leq r(\Psi_1 \cdots \Psi_m)^\alpha, \end{aligned} \quad (2.37)$$

$$\begin{aligned} r(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}) &\leq r(\Psi_1^{(\alpha m)} \cdots \Psi_m^{(\alpha m)})^{\frac{1}{m}} \leq \\ r((\Psi_1 \cdots \Psi_m)^{(\alpha m)})^{\frac{1}{m}} &\leq r(((\Psi_1 \cdots \Psi_m)^n)^{(\alpha m)})^{\frac{1}{nm}} \leq r(\Psi_1 \cdots \Psi_m)^\alpha. \end{aligned} \quad (2.38)$$

If, in addition, $\alpha \geq 1$ then

$$\begin{aligned} r(\Psi_1^{(\alpha)} \circ \cdots \circ \Psi_m^{(\alpha)}) &\leq r(\Phi_1^{(\alpha)} \circ \cdots \circ \Phi_m^{(\alpha)})^{\frac{1}{m}} \leq r((\Phi_1^n)^{(\alpha)} \circ \cdots \circ (\Phi_m^n)^{(\alpha)})^{\frac{1}{mn}} \\ &\leq (r((\Phi_1^n)^{(m)})) \cdots r((\Phi_m^n)^{(m)})^{\frac{\alpha}{m^2 n}} \leq r(\Psi_1 \cdots \Psi_m)^\alpha, \end{aligned} \quad (2.39)$$

$$\begin{aligned}
 r\left(\Psi_1^{(\alpha)} \circ \dots \circ \Psi_m^{(\alpha)}\right) &\leq r\left(\Sigma_1^{(\frac{1}{m})} \circ \dots \circ \Sigma_m^{(\frac{1}{m})}\right)^{\frac{1}{m}} \\
 &\leq r\left((\Sigma_1^n)^{(\frac{1}{m})} \circ \dots \circ (\Sigma_m^n)^{(\frac{1}{m})}\right)^{\frac{1}{mn}} \leq r\left(\Psi_1^{(\alpha m)} \dots \Psi_m^{(\alpha m)}\right)^{\frac{1}{m}} \\
 &\leq r\left((\Psi_1 \dots \Psi_m)^{(\alpha m)}\right)^{\frac{1}{m}} \leq r\left(((\Psi_1 \dots \Psi_m)^n)^{(\alpha m)}\right)^{\frac{1}{nm}} \leq r(\Psi_1 \dots \Psi_m)^\alpha. \quad (2.40)
 \end{aligned}$$

Proof: Inequalities (2.37) are proved in a similar way as Theorem 2.3.3 by applying Theorems 1.4.3 and 2.3.2(ii) instead of Theorems 1.4.1 and 2.3.2(i). For the proof of (2.38) observe that

$$\Psi_1^{(\alpha)} \circ \dots \circ \Psi_m^{(\alpha)} = (\Psi_1^{(\alpha m)})^{(\frac{1}{m})} \circ \dots \circ (\Psi_m^{(\alpha m)})^{(\frac{1}{m})}$$

for $i = 1, \dots, m$. Now the first inequality in (2.38) follows from (1.64) (or from (2.39)):

$$\begin{aligned}
 r\left(\Psi_1^{(\alpha)} \circ \dots \circ \Psi_m^{(\alpha)}\right) &= r\left((\Psi_1^{(\alpha m)})^{(\frac{1}{m})} \circ \dots \circ (\Psi_m^{(\alpha m)})^{(\frac{1}{m})}\right) \\
 &\leq r\left(\Psi_1^{(\alpha m)} \dots \Psi_m^{(\alpha m)}\right)^{\frac{1}{m}}.
 \end{aligned}$$

Other inequalities in (2.38) follow from Theorem 2.3.2(ii).

Assume $\alpha \geq 1$. The first and second inequality in (2.39) follow from (2.37). To prove the third inequality in (2.39) notice that $(\Phi_i^n)^{(\alpha)} = ((\Phi_i^n)^{(m)})^{(\frac{\alpha}{m})}$, $\frac{\alpha}{m} \geq \frac{1}{m}$ and apply Theorem 1.4.3. The last inequality in (2.39) follows again from Theorem 1.4.3 and the fact that $r(\Phi_1) = \dots = r(\Phi_m) = r(\Psi_1 \dots \Psi_m)$.

To prove the first three inequalities in (2.40) observe that $\Psi_i^{(\alpha)} = (\Psi_i^{(\alpha m)})^{(\frac{1}{m})}$, $\frac{\alpha}{m} \geq \frac{1}{m}$ and apply Theorem 2.3.3. The remaining three inequalities in (2.40) follow from (2.38), which completes the proof. $\blacksquare$

The following result was established in [6].

Theorem 2.3.6 *Let $\{\Psi_{ij}\}_{i=1, j=1}^{k, m}$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \dots, \alpha_m$ be positive numbers.*

(i) *If $r \in \{\rho, \hat{\rho}\}$, $\sum_{j=1}^m \alpha_j = 1$ and $n \in \mathbb{N}$, then*

$$\begin{aligned}
 &r\left(\left(\Psi_{11}^{(\alpha_1)} \circ \dots \circ \Psi_{1m}^{(\alpha_m)}\right) + \dots + \left(\Psi_{k1}^{(\alpha_1)} \circ \dots \circ \Psi_{km}^{(\alpha_m)}\right)\right) \\
 &\leq r\left((\Psi_{11} + \dots + \Psi_{k1})^{(\alpha_1)} \circ \dots \circ (\Psi_{1m} + \dots + \Psi_{km})^{(\alpha_m)}\right) \\
 &\leq r\left(((\Psi_{11} + \dots + \Psi_{k1})^n)^{(\alpha_1)} \circ \dots \circ ((\Psi_{1m} + \dots + \Psi_{km})^n)^{(\alpha_m)}\right)^{\frac{1}{n}} \\
 &\leq r(\Psi_{11} + \dots + \Psi_{k1})^{\alpha_1} \dots r(\Psi_{1m} + \dots + \Psi_{km})^{\alpha_m}. \quad (2.41)
 \end{aligned}$$

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If, in addition, L and L^* have order continuous norms, then Inequalities (2.41) hold also for each $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$.

(ii) If $L \in \mathcal{L}$, $r \in \{\rho, \hat{\rho}\}$, $\sum_{j=1}^m \alpha_j \geq 1$ and $\{\Psi_{ij}\}_{i=1, j=1}^{k, m}$ are bounded sets of nonnegative matrices that define positive operators on L , then Inequalities (2.41) hold.

Proof: (i) Let $r \in \{\rho, \hat{\rho}\}$, $\sum_{i=1}^m \alpha_i = 1$ and $n \in \mathbb{N}$. To prove the first inequality in (2.41) let $l \in \mathbb{N}$ and

$$A \in \left(\left(\Psi_{11}^{(\alpha_1)} \circ \dots \circ \Psi_{1m}^{(\alpha_m)} \right) + \dots + \left(\Psi_{k1}^{(\alpha_1)} \circ \dots \circ \Psi_{km}^{(\alpha_m)} \right) \right)^l.$$

Then $A = A_1 \cdots A_l$, where for each $i = 1, \dots, l$ we have

$$A_i = \left(A_{i11}^{(\alpha_1)} \circ \dots \circ A_{i1m}^{(\alpha_m)} \right) + \dots + \left(A_{ik1}^{(\alpha_1)} \circ \dots \circ A_{ikm}^{(\alpha_m)} \right),$$

where $A_{i11} \in \Psi_{11}, \dots, A_{i1m} \in \Psi_{1m}, \dots, A_{ik1} \in \Psi_{k1}, \dots, A_{ikm} \in \Psi_{km}$. Then by (1.57) for each $i = 1, \dots, l$ we have

$$A_i \leq C_i := (A_{i11} + A_{i21} + \dots + A_{ik1})^{(\alpha_1)} \circ \dots \circ (A_{i1m} + A_{i2m} + \dots + A_{ikm})^{(\alpha_m)},$$

where $C_i \in (\Psi_{11} + \dots + \Psi_{k1})^{(\alpha_1)} \circ \dots \circ (\Psi_{1m} + \dots + \Psi_{km})^{(\alpha_m)}$. Therefore

$$A \leq C := C_1 \cdots C_l \in ((\Psi_{11} + \dots + \Psi_{k1})^{(\alpha_1)} \circ \dots \circ (\Psi_{1m} + \dots + \Psi_{km})^{(\alpha_m)})^l,$$

$r(A)^{1/l} \leq r(C)^{1/l}$ and $\|A\|^{1/l} \leq \|C\|^{1/l}$, which implies the first inequality in (2.41). The second and third inequality in (2.41) follow from (1.63).

If, in addition, L and L^* have order continuous norms and $r \in \{\rho_{ess}, \hat{\rho}_{ess}\}$, then Inequalities (2.41) are proved similarly. Under the assumptions of (ii) Inequalities (2.41) are proved in a similar way by applying Theorems 1.2.16(ii) and 1.4.3. ■

Let Ψ and Σ be bounded sets of positive kernel operators on $L^2(X, \mu)$ and $\alpha \in [0, 1]$. Denote by Ψ^* and $S_\alpha(\Psi)$ bounded sets of positive kernel operators on $L^2(X, \mu)$ defined by $\Psi^* = \{A^* : A \in \Psi\}$ and

$$S_\alpha(\Psi) = \Psi^{(\alpha)} \circ (\Psi^*)^{(1-\alpha)} = \{A^{(\alpha)} \circ (B^*)^{(1-\alpha)} : A, B \in \Psi\}.$$

We denote simply $S(\Psi) = S_{\frac{1}{2}}(\Psi)$, the geometric symmetrization of Ψ . Observe that $(\Psi\Sigma)^* = \Sigma^*\Psi^*$ and $(\Psi^m)^* = (\Psi^*)^m$ for all $m \in \mathbb{N}$. By (1.63) it follows that

$$r(S_\alpha(\Psi)) \leq r(S_\alpha(\Psi^m))^{\frac{1}{m}} \leq r(\Psi) \quad (2.42)$$

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for all $m \in \mathbb{N}$ and $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$, since $r(\Psi) = r(\Psi^*)$. In particular, for all $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$ and $n \in \mathbb{N} \cup \{0\}$ we have

$$r(S_\alpha(\Psi)) \leq r(S_\alpha(\Psi^{2^n}))^{2^{-n}} \leq r(\Psi). \quad (2.43)$$

Consequently,

$$r(S_\alpha(\Psi))^2 \leq r(S_\alpha(\Psi^2)) \leq r(\Psi)^2 \quad (2.44)$$

holds for all $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$.

The following result that follows from (2.43) was established in [6], is an extension of Theorem 2.1.3 and extends [9, Theorem 2.2], [39, Theorem 3.5] and [5, Theorem 3.5(ii)].

Theorem 2.3.7 *Let Ψ be a bounded set of positive kernel operators on $L^2(X, \mu)$, $\alpha \in [0, 1]$ and let $r_n = r(S_\alpha(\Psi^{2^n}))^{2^{-n}}$ for $n \in \mathbb{N} \cup \{0\}$ and $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$. Then for each n*

$$r(S_\alpha(\Psi)) = r_0 \leq r_1 \leq \cdots \leq r_n \leq r(\Psi).$$

Proof: By (2.43) we have $r_n \leq r(\Psi)$. Since $r_{n-1} \leq r_n$ for all $n \in \mathbb{N}$ by the first inequality in (2.44), the proof is completed. ■

The following result, established in [6], extends Proposition 2.1.1.

Proposition 2.3.8 *Let $\Psi_1, \dots, \Psi_m$ be bounded sets of positive kernel operators on $L^2(X, \mu)$, $\alpha \in [0, 1]$, $n \in \mathbb{N}$ and $r \in \{\rho, \hat{\rho}, \rho_{ess}, \hat{\rho}_{ess}\}$. Then we have*

$$\begin{aligned} r(S_\alpha(\Psi_1) \cdots S_\alpha(\Psi_m)) &\leq r((\Psi_1 \cdots \Psi_m)^{(\alpha)} \circ ((\Psi_m \cdots \Psi_1)^*)^{(1-\alpha)}) \\ &\leq r(((\Psi_1 \cdots \Psi_m)^n)^{(\alpha)} \circ (((\Psi_m \cdots \Psi_1)^*)^n)^{(1-\alpha)})^{\frac{1}{n}} \\ &\leq r(\Psi_1 \cdots \Psi_m)^\alpha r(\Psi_m \cdots \Psi_1)^{1-\alpha}, \end{aligned} \quad (2.45)$$

$$\begin{aligned} r(S_\alpha(\Psi_1) + \cdots + S_\alpha(\Psi_m)) &\leq r(S_\alpha(\Psi_1 + \cdots + \Psi_m)) \\ &\leq r(S_\alpha((\Psi_1 + \cdots + \Psi_m)^n))^{\frac{1}{n}} \leq r(\Psi_1 + \cdots + \Psi_m). \end{aligned} \quad (2.46)$$

In particular, we have

$$\begin{aligned} r(S_\alpha(\Psi_1)S_\alpha(\Psi_2)) &\leq r((\Psi_1\Psi_2)^{(\alpha)} \circ ((\Psi_2\Psi_1)^*)^{(1-\alpha)}) \\ &\leq r(((\Psi_1\Psi_2)^n)^{(\alpha)} \circ (((\Psi_2\Psi_1)^*)^n)^{(1-\alpha)})^{\frac{1}{n}} \leq r(\Psi_1\Psi_2). \end{aligned} \quad (2.47)$$

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Proof: By Theorem 2.3.2(i) we have

$$\begin{aligned}
 r(S_\alpha(\Psi_1) \cdots S_\alpha(\Psi_m)) &= r\left((\Psi_1^{(\alpha)} \circ (\Psi_1^*)^{(1-\alpha)}) \cdots (\Psi_m^{(\alpha)} \circ (\Psi_m^*)^{(1-\alpha)})\right) \\
 &\leq r\left((\Psi_1 \cdots \Psi_m)^{(\alpha)} \circ ((\Psi_m \cdots \Psi_1)^*)^{(1-\alpha)}\right) \\
 &\leq r\left(((\Psi_1 \cdots \Psi_m)^n)^{(\alpha)} \circ (((\Psi_m \cdots \Psi_1)^*)^n)^{(1-\alpha)}\right)^{\frac{1}{n}} \\
 &\leq r(\Psi_1 \cdots \Psi_m)^\alpha r((\Psi_m \cdots \Psi_1)^*)^{1-\alpha} = r(\Psi_1 \cdots \Psi_m)^\alpha r(\Psi_m \cdots \Psi_1)^{1-\alpha},
 \end{aligned}$$

where the last equality follows from the fact that $r(\Psi) = r(\Psi^*)$. The inequalities in (2.46) are proved in similar way by applying Theorem 2.3.6 and (2.43). The first and second inequalities in (2.47) are special cases of (2.45), while the third inequality follows from (2.45) and the fact that $r(\Psi_1\Psi_2) = r(\Psi_2\Psi_1)$. ■

Let Ψ be a bounded set of nonnegative matrices that define operators on $l^2(R)$ and let α and β be nonnegative numbers such that $\alpha + \beta \geq 1$. The set $S_{\alpha,\beta}(\Psi) = \Psi^{(\alpha)} \circ (\Psi^*)^{(\beta)} = \{A^{(\alpha)} \circ (B^*)^{(\beta)} : A, B \in \Psi\}$ is a bounded set of nonnegative matrices that define operators on $l^2(R)$ by Theorem 1.2.16(ii).

For $r \in \{\rho, \hat{\rho}\}$ the following result extends Theorem 2.3.7 in the case of bounded set of nonnegative matrices that define operators on $l^2(R)$. It also extends a part of [5, Theorem 3.5(ii)] and was established in [6].

Theorem 2.3.9 *Let Ψ be a bounded set of nonnegative matrices that define operators on $l^2(R)$ and $r \in \{\rho, \hat{\rho}\}$. Assume α and β are nonnegative numbers such that $\alpha + \beta \geq 1$ and denote $r_n = r(S_{\alpha,\beta}(\Psi^{2^n}))^{2^{-n}}$ for $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$r(S_{\alpha,\beta}(\Psi)) = r_0 \leq r_1 \leq \cdots \leq r_n \leq r(\Psi)^{\alpha+\beta}. \quad (2.48)$$

Proof: By Theorem 1.4.3 we have

$$r(S_{\alpha,\beta}(\Psi)) = r(\Psi^{(\alpha)} \circ (\Psi^*)^{(\beta)}) \leq r((\Psi^{2^n})^{(\alpha)} \circ ((\Psi^*)^{2^n})^{(\beta)})^{2^{-n}} = r_n \leq r(\Psi)^{\alpha+\beta}. \quad (2.49)$$

In particular, for $n = 1$ we have

$$r(S_{\alpha,\beta}(\Psi))^2 \leq r(S_{\alpha,\beta}(\Psi^2)) \leq r(\Psi)^{2(\alpha+\beta)}. \quad (2.50)$$

Since $r_{n-1} \leq r_n$ for all $n \in \mathbb{N} \cup \{0\}$ by the first inequality in (2.50), the proof of (2.48) is completed. ■

The following result, established in [6], is proved in similar way as Proposition 2.3.8 using Theorem 2.3.2(ii) instead of Theorem 2.3.2(i).

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Proposition 2.3.10 *Let $\Psi, \Psi_1, \dots, \Psi_m$ be bounded sets of nonnegative matrices that define operators on $l^2(R)$, $n \in \mathbb{N}$ and let α and β be nonnegative numbers such that $\alpha + \beta \geq 1$. Then we have*

$$\begin{aligned} r(S_{\alpha,\beta}(\Psi_1) \cdots S_{\alpha,\beta}(\Psi_m)) &\leq r((\Psi_1 \cdots \Psi_m)^{(\alpha)} \circ ((\Psi_m \cdots \Psi_1)^*)^{(\beta)}) \\ &\leq r(((\Psi_1 \cdots \Psi_m)^n)^{(\alpha)} \circ (((\Psi_m \cdots \Psi_1)^*)^n)^{(\beta)})^{\frac{1}{n}} \\ &\leq r(\Psi_1 \cdots \Psi_m)^\alpha r(\Psi_m \cdots \Psi_1)^\beta, \end{aligned} \quad (2.51)$$

$$r(S_{\alpha,\beta}(\Psi)) \leq r(S_{\alpha,\beta}(\Psi^n))^{\frac{1}{n}} \leq r(\Psi)^{\alpha+\beta}, \quad (2.52)$$

$$\begin{aligned} r(S_{\alpha,\beta}(\Psi_1) + \cdots + S_{\alpha,\beta}(\Psi_m)) &\leq r(S_{\alpha,\beta}(\Psi_1 + \cdots + \Psi_m)) \\ &\leq r(S_{\alpha,\beta}((\Psi_1 + \cdots + \Psi_m)^n))^{\frac{1}{n}} \leq r(\Psi_1 + \cdots + \Psi_m)^{\alpha+\beta}, \end{aligned} \quad (2.53)$$

$$\begin{aligned} r(S_{\alpha,\beta}(\Psi_1) + \cdots + S_{\alpha,\beta}(\Psi_m)) &\leq r(S_{\alpha,\beta}(\Psi_1 + \cdots + \Psi_m)) \\ &\leq r(S_{\alpha,\beta}((\Psi_1 + \cdots + \Psi_m)^n))^{\frac{1}{n}} \leq r(\Psi_1 + \cdots + \Psi_m)^{\alpha+\beta}, \end{aligned} \quad (2.54)$$

$$\begin{aligned} r(S_{\alpha,\beta}(\Psi_1)S_{\alpha,\beta}(\Psi_2)) &\leq r((\Psi_1\Psi_2)^{(\alpha)} \circ ((\Psi_2\Psi_1)^*)^{(\beta)}) \\ &\leq r(((\Psi_1\Psi_2)^n)^{(\alpha)} \circ (((\Psi_2\Psi_1)^*)^n)^{(\beta)})^{\frac{1}{n}} \leq r(\Psi_1\Psi_2)^{\alpha+\beta} \end{aligned} \quad (2.55)$$

for $r \in \{\rho, \hat{\rho}\}$.

Proof: Inequalities (2.51) and (2.54) are proved in a similar way as inequalities (2.45) and (2.46) by using Theorems 2.3.2(ii) and 2.3.6(ii). Inequalities (2.52) and (2.55) are special cases of (2.51). ■

Remark 2.3.11 *The obtained results remain valid when spectral radius ρ is replaced by the essential spectral radius ρ_{ess} and the operator norm $\|\cdot\|$ by the Hausdorff measure of non-compactness γ . To be more precise, the essential versions of results Theorem 2.2.4(i), Corollary 2.2.5(i), Theorem 2.2.8(i), Corollary 2.2.9(i), Corollary 2.2.10(i) hold on $L^2(X, \mu)$ and the essential versions of results Theorem 2.2.4(ii), Corollary 2.2.5(ii), Theorem 2.2.6, Corollary 2.2.7, Theorem 2.2.8(ii), Corollary 2.2.9(ii), Corollary 2.2.10(ii), Lemma 2.2.11, Corollary 2.2.12, Theorem 2.3.9 and Proposition 2.3.10 hold*

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on $l^2(R)$, while the essential versions of the results Theorem 2.2.3, Theorem 2.3.2(ii), Theorem 2.3.5 and Theorem 2.3.6(ii) hold when L and L^* have order continuous norms. These facts were proved in [28] by applying Lemma 1.3.7 and Theorem 1.3.3 in the setting of bounded sets of positive kernel operators.

Chapter 3

Relations of Schur-multipliers to derivation's inequalities of operators

3.1 Symmetrically norming functions and associated ideals of compact operators

Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}_\infty(\mathcal{H})$ denote respectively spaces of all bounded and all compact linear operators acting on a separable, complex space $\mathcal{H}$.

Definition 3.1.1 *Let $\mathfrak{c}_0$ be the space of all sequences $\xi = \{\xi_n\}_{n=1}^\infty$ of real numbers which tend to zero. We denote by $\hat{\mathfrak{c}}$ the lineal of $\mathfrak{c}_0$, consisting of all sequences with a finite number of nonzero terms. A real function $\Phi(\xi) = \Phi(\xi_1, \xi_2, \dots)$ defined on the lineal $\hat{\mathfrak{c}}$ is called a symmetrically norming (s.n) function if the following properties are satisfied:*

- i) $\Phi(\xi) > 0$ ($\xi \in \hat{\mathfrak{c}}, \xi \neq 0$),*
- ii) for any real α $\Phi(\alpha\xi) = |\alpha|\Phi(\xi)$ ($\xi \in \hat{\mathfrak{c}}$),*
- iii) $\Phi(\xi + \eta) \leq \Phi(\xi) + \Phi(\eta)$ ($\xi, \eta \in \hat{\mathfrak{c}}$),*
- iv) $\Phi(1, 0, \dots) = 1$*
- v) $\Phi(\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots) = \Phi(|\xi_{j_1}|, |\xi_{j_2}|, \dots, |\xi_{j_n}|, 0, 0, \dots)$,*

where $\xi = \{\xi_n\}_{n=1}^\infty$ is any vector from $\hat{\mathfrak{c}}$ and $j_1, j_2, \dots, j_n$ is any permutation of integers $1, 2, \dots, n$.

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We can extend the domain of the function $\Phi(\xi)$ on $\mathbf{c}_\Phi$, where

$$\mathbf{c}_\Phi \stackrel{\text{def}}{=} \left\{ \xi \in \mathbf{c}_0 \mid \sup_{n \geq 1} \Phi(\xi_1, \dots, \xi_n, 0, 0, \dots) < +\infty \right\}$$

and define $\Phi(\xi) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \Phi(\xi_1, \dots, \xi_n, 0, 0, \dots)$ for any $\xi \in \mathbf{c}_\Phi$. The limes on $\mathbf{c}_\Phi$ is well defined since the sequence $\{\Phi(\xi_1, \dots, \xi_n, 0, 0, \dots)\}_{n=1}^\infty$ is non-decreasing.

So, each „symmetrically norming” (s.n.) function, or also called „symmetric gauge” function Φ on sequences of complex numbers, gives rise to a symmetric or unitarily invariant (u.i.) norm on compact operators defined by $\|A\|_\Phi \stackrel{\text{def}}{=} \Phi(\{s_n(A)\}_{n=1}^\infty)$, where $s_1(A) \geq s_2(A) \geq \dots$ are the singular values of A , i.e., the eigenvalues of $|A| \stackrel{\text{def}}{=} (A^*A)^{1/2}$. Any such norm is defined on the naturally associated norm ideal $\mathbf{C}_\Phi(\mathcal{H})$ of $\mathbf{C}_\infty(\mathcal{H})$, defined as a set of all operators $A \in \mathbf{C}_\Phi(\mathcal{H})$ for which $s(A) = \{s_n(A)\}_{n \in \mathbb{N}} \in \mathbf{c}_\Phi$ and it satisfies the invariance property $\|UAV\|_\Phi = \|A\|_\Phi$ for all $A \in \mathbf{C}_\Phi(\mathcal{H})$ and for all unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ (see [14] and [42] for more details). Examples of u.i. norms are Schatten p -norms defined by $\|A\|_p \stackrel{\text{def}}{=} (\sum_{n=1}^\infty s_n^p(A))^{1/p}$, for $1 \leq p < +\infty$, while $\|A\|_\infty \stackrel{\text{def}}{=} s_1(A)$ coincides with the operator norm $\|A\|$. For $p := 1$ the corresponding s.n. function is the trace s.n. function (also denoted by ℓ^1 or ℓ_1), defined by $\ell((\lambda_n)_{n=1}^\infty) = \sum_{n=1}^\infty |\lambda_n|$, while for $p = \infty$ the s.n. function ℓ^∞ is defined by $\ell^\infty((\lambda_n)_{n=1}^\infty) = \sup_{n \in \mathbb{N}} |\lambda_n|$. The $\mathbf{C}_1(\mathcal{H})$ class is known as the trace class or the class of nuclear operators, while $\mathbf{C}_2(\mathcal{H})$ is known as the Hilbert-Schmidt class. Ideals of compact operators associated to these norms will be denoted by $\mathbf{C}_p(\mathcal{H})$. Schatten p -norms represent basic examples of (by the degree) p modified norms. Namely, for any $p > 0$, a u.i. norm $\|\cdot\|_\Phi$ can be p -modified by setting $\|A\|_{\Phi^{(p)}} \stackrel{\text{def}}{=} \||A|^p\|_\Phi^{1/p}$, for all $A \in \mathbf{C}_\infty(\mathcal{H})$ such that $|A|^p \in \mathbf{C}_\Phi(\mathcal{H})$. We refer to a s.n. function $\Phi^{(p)}$ as to a p -modified function. For the simple proof of the triangle inequality (for $p \geq 1$) and other properties of these norms, including Hölder's inequality, see the preliminary section in [19], as well as Corollary IV.2.6 and Exercises IV.2.7-8 in [2]. If $p := 2$, then $\|\cdot\|_{\Phi^{(2)}}$ are also known as Q -norms. Hence, as $\Phi^{(p)} = (\Phi^{(\frac{p}{2})})^{(2)}$, then $\Phi^{(p)}$ are also Q -norms for all $p \geq 2$, while its dual norms $\|\cdot\|_{\Phi^{(p)}^*}$ are commonly known as Q^* -norms. Each norm $\|\cdot\|_\Phi$ is lower semi-continuous, i.e., $\|w\text{-}\lim_{n \rightarrow \infty} A_n\|_\Phi \leq \liminf_{n \rightarrow \infty} \|A_n\|_\Phi$. This follows from the uniform boundedness principle and the well known representation

formula

$$\|A\|_{\Phi} = \sup \left\{ \frac{|\operatorname{tr}(AB)|}{\|B\|_{\Phi^*}} : B \text{ is a finite rank operator,} \right\}$$

where Φ^* stands for the s.n. function dual to Φ (see [42, Th. 2.7 (d)]). Monotonicity is another useful property of all u.i. norms, saying that $s_n(A) \leq s_n(B)$, for all $n \in \mathbb{N}$ implies $\|A\|_{\Phi} \leq \|B\|_{\Phi}$. This may be combined with the monotonicity of singular numbers, which states that $s_n(A) \leq s_n(B)$, for all $n \in \mathbb{N}$, whenever $0 \leq A \leq B$. Moreover, we have the following double monotonicity property for u.i. norms, saying that

$$\|AXB\|_{\Phi} \leq \|CXD\|_{\Phi} \quad (3.1)$$

whenever $A^*A \leq C^*C$ and $BB^* \leq DD^*$. For the proof of (3.1) see [22, p.62].

An operator $A \in \mathcal{B}(\mathcal{H})$ is called hyponormal if and only iff $A^*A \geq AA^*$, and similarly, A is cohyponormal iff A^* is hyponormal, i.e., iff $AA^* \geq A^*A$. Also $A \in \mathcal{B}(\mathcal{H})$ is called accretive iff $A_{\Re} \stackrel{\text{def}}{=} \frac{A+A^*}{2} \geq 0$ and strictly accretive iff $A_{\Re} \geq cI$ for some $c > 0$.

For a more complete account of the theory of norm ideals, the reader is referred to [2], [14], [13], [40] and [42].

3.2 Gel'fand or weak*-integral

If $(\Omega, \mathfrak{M}, \mu)$ is a space Ω with a measure μ on σ -algebra $\mathfrak{M}$, then we will refer to a function $A: \Omega \mapsto \mathcal{B}(\mathcal{H}): t \mapsto A_t$ as to a weakly*-measurable if $t \mapsto \langle A_t g, h \rangle$ is measurable for all $g, h \in \mathcal{H}$. If, in addition, those functions are integrable, then there is the unique (known as Gel'fand or weak*-integral and denoted by $\int_{\Omega} A_t d\mu(t)$) operator in $\mathcal{B}(\mathcal{H})$, satisfying

$$\left\langle \int_{\Omega} A_t d\mu(t) h, k \right\rangle = \int_{\Omega} \langle A_t h, k \rangle d\mu(t) \quad \text{for all } h, k \in \mathcal{H}. \quad (3.2)$$

Thus, it also complies with the definition of Pettis integral. For a more complete account about weak*-integrals the reader is referred to [8, p.53], [18, p.320] and [21, Lemma 1.2]. For every $h \in \mathcal{H}$, the function $t \mapsto \|A_t h\|$ is also measurable, and, if additionally $\int_{\Omega} \|A_t h\|^2 d\mu(t) < +\infty$ for

all $h \in \mathcal{H}$, then there exists weak*-integral $\int_{\Omega} A_t^* A_t d\mu(t) \in \mathcal{B}(\mathcal{H})$, satisfying $\left\langle \int_{\Omega} A_t^* A_t d\mu(t) h, h \right\rangle = \int_{\Omega} \|A_t h\|^2 d\mu(t)$ for all $h \in \mathcal{H}$, as shown in [18, ex.2]. Such families $\{A_t\}_{t \in \Omega}$ will be simple called $[\mu]$ square integrable ($[\mu]$ s.i.).

We use the notation $L^2(\Omega, \mu, \mathcal{H})$ for the space of all (weakly) measurable functions $f : \Omega \mapsto \mathcal{H}$ such that $\int_{\Omega} \|f(t)\|^2 d\mu(t) < +\infty$ and notation $L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ for the space of all weak*-measurable functions $A : \Omega \mapsto \mathcal{B}(\mathcal{H})$ such that $\int_{\Omega} \|A_t h\|^2 d\mu(t) < +\infty$ for all $h \in \mathcal{H}$, i.e. for $[\mu]$ s.i. families. Thus $A \in L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ if and only if $Af \in L^2(\Omega, \mu, \mathcal{H})$ for all $f \in \mathcal{H}$.

3.3 Operator monotone functions

Definition 3.3.1 *A real function φ defined on an interval $J \subset \mathbb{R}$ is said to be matrix monotone of order n if for every pair of $n \times n$ hermitian matrices A, B with $\sigma(A), \sigma(B) \subset J$ we have $A \leq B \Rightarrow \varphi(A) \leq \varphi(B)$. If φ is a matrix monotone function of order n for all $n \in \mathbb{N}$ (or the above property holds for every $A, B \in \mathcal{B}(\mathcal{H})$ with arbitrary Hilbert space $\mathcal{H}$), then φ is said to be operator monotone.*

We are going to consider a more general setting, when $\mathcal{H}$ is a complex, separable, Hilbert space and $A, B \in \mathcal{B}(\mathcal{H})$.

Example 3.3.2 (1) When $a \geq 0$ the function $t \mapsto at + b$ is operator monotone on $\mathbb{R}$.

(2) When $c \notin (a, b)$ the function $t \mapsto (c - t)^{-1}$ is operator monotone on (a, b) .

(3) When $0 \leq p \leq 1$ the function $t \mapsto t^p$ is operator monotone on $[0, +\infty)$.

(4) $t \mapsto \frac{t}{t+1}$ is operator monotone on $[0, +\infty)$.

(5) $t \mapsto \log t$ is operator monotone on $(0, +\infty)$.

(6) $f(t) := \frac{t-1}{\log t}$ is operator monotone on $[0, +\infty)$, where $f(0) = 0$ and $f(1) = 1$.

The study of operator monotone functions was introduced by Löwner. His classical theorem states that operator monotone functions on real intervals are described by holomorphic functions on the upper half plane. K. Löwner in [29] gave a definite characterization of operator monotone functions in 1934. His well known theorem states that function $\varphi : (a, b) \mapsto \mathbb{R}$ is operator

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monotone if and only if φ has analytic continuation to $(\mathbb{C} \setminus \mathbb{R} \cup (a, b))$ which maps the upper half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \Im z > 0\}$ into itself and the extension to the lower half plane $\mathbb{C}^- := \{z \in \mathbb{C} : \Im z < 0\}$ is obtained by reflection across (a, b) , that is $\varphi(\bar{z}) = \overline{\varphi(z)}$ for all $z \in \mathbb{C}^+$.

The following theorem gives the integral representation of an operator monotone function on $(-1, 1)$. See [16, Theorem 2.5.7] for the proof.

Theorem 3.3.3 *Let f be a non-constant operator monotone function on $(-1, 1)$. Then, there exists a unique probability Borel measure μ on $[-1, 1]$ such that*

$$f(x) = f(0) + f'(0) \int_{-1}^1 \frac{x}{1 - \lambda x} d\mu(\lambda), \quad x \in (-1, 1). \quad (3.3)$$

The previous result can be transferred to an arbitrary finite interval (a, b) since an arbitrary function f on is operator monotone on (a, b) if and only if $f\left(\frac{(b-a)t}{2} + \frac{a+b}{2}\right)$ is operator monotone on $(-1, 1)$.

For continuous positive functions defined on $[0, +\infty)$ we have the following integral representation. For the proof see e. g. [16].

Theorem 3.3.4 *Let f be a continuous non-negative function on $[0, +\infty)$. Then f is operator monotone if and only if there exists a positive finite Borel measure m on $[0, +\infty]$ such that*

$$f(t) = \int_{[0, \infty]} \frac{t(1 + \lambda)}{t + \lambda} dm(\lambda), \quad t \in [0, \infty).$$

The measure m is unique, and if $a := m(\{0\})$ and $b := m(\{\infty\})$ then

$$f(t) = a + bt + \int_{(0, \infty)} \frac{t(1 + \lambda)}{t + \lambda} dm(\lambda), \quad t \in [0, \infty). \quad (3.4)$$

Also, $a = f(0)$ and $b = \lim_{t \rightarrow \infty} \frac{f(t)}{t}$. Moreover, a continuous real-valued function f on $[0, \infty)$ is operator monotone if and only if there exist $a \geq 0$ and a positive Borel measure m on $(0, \infty)$ such that (3.4) is valid.

Proof: From $\frac{t(1+\lambda)}{t+\lambda} = 1 + \lambda - \frac{\lambda(1+\lambda)}{t+\lambda}$ for every $\lambda \in [0, \infty)$ it follows that $\frac{t(1+\lambda)}{t+\lambda}$ is operator monotone on $[0, \infty)$ and therefore f is also an operator monotone

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function if the above integral representation holds. Conversely, assume that $f \geq 0$ is continuous and operator monotone on $[0, \infty)$. Consider the function $\psi(x) = \frac{1+x}{1-x}$ on $(-1, 1)$. It is a bijective mapping between intervals $(-1, 1)$ and $(0, \infty)$ and since $t = \psi(x) = \frac{1+x}{1-x} = -1 + \frac{2}{1-x}$, ψ is an operator monotone function on $(-1, 1)$. It follows that $g(x) := f(\psi(x))$ is also an operator monotone function on $(-1, 1)$. Therefore, Theorem 3.3.3 implies that there exists a probability Borel measure μ on $[-1, 1]$ such that

$$g(x) = g(0) + g'(0) \int_{[-1, 1]} \frac{x}{1 - \lambda x} d\mu(\lambda), \quad x \in (-1, 1).$$

Since $g(-1) = \lim_{x \rightarrow -1+} g(x) = f(0) \geq 0$ it follows that $\int_{[-1, 1]} \frac{1}{1+\lambda} d\mu(\lambda) < \infty$ and in particular $\mu(\{-1\}) = 0$. Hence,

$$g(x) - g(-1) = g'(0) \int_{(-1, 1]} \frac{1+x}{(1+\lambda x)(1+\lambda)} d\mu(\lambda).$$

Transforming this by $x = \psi^{-1}(t)$ and $\lambda = \psi^{-1}(\zeta)$ and introducing the measure m on $(0, \infty]$ by $m \stackrel{\text{def}}{=} \tilde{\mu} \circ \psi^{-1}$, where $d\tilde{\mu}(\lambda) := \frac{g'(0)}{1+\lambda} d\mu(\lambda)$ we obtain

$$f(t) - f(0) = \int_{(0, \infty]} \frac{t(\zeta + 1)}{\zeta + t} dm(\zeta), \quad t \in [0, \infty).$$

Adding the mass $f(0)\delta_0(\{0\})$ to m , where δ_0 is Dirac measure we have

$$f(t) = \int_{[0, \infty]} \frac{t(\zeta + 1)}{\zeta + t} dm(\zeta), \quad t \in [0, \infty].$$

The uniqueness of the measure m follows from the Theorem 3.3.3. The last statement follows by applying the above to the case $f - f(0)$. ■

For $J := [0, +\infty)$, transforming the measure in Theorem 3.3.4 we obtain the following integral representation theorem, which we will use in the sequel.

Theorem 3.3.5 *A function $f : [0, +\infty) \mapsto \mathbb{R}$ is operator monotone if and only if there is $a \in \mathbb{R}$, $b \geq 0$ and a positive Borel measure μ on $(0, +\infty)$ satisfying $\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < +\infty$, such that*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda) \tag{3.5}$$

Note also that $\varphi(0) = a$.

For some more details about operator monotone functions see [2, p.p. 144-145] and [16, Theorem 2.7.11].

3.4 Spectral measure, integration and functional calculus

In this section we provide an overview of projection valued or spectral measure, integrals with respect of such measures and functional calculus for selfadjoint and normal operators. For more details see e.g. [3].

Denote by $\mathcal{P}(\mathcal{H})$ the family of all orthogonal projections on $\mathcal{H}$ and let $(\Omega, \mathfrak{M})$ be an arbitrary measurable space.

Definition 3.4.1 *A mapping $E : \mathfrak{M} \mapsto \mathcal{P}(\mathcal{H})$ is called a spectral measure on $\mathcal{H}$ if the following conditions are satisfied:*

- i) $E(\bigcup_{n=1}^{\infty} \delta_n) = s - \sum_{n=1}^{\infty} E(\delta_n)$, where $\{\delta_n\}_{n=1}^{\infty}$ is countable or finite set of disjoint sets $\delta_n \in \mathfrak{M}$ (countable additivity).
- ii) $E(\Omega) = I$ (completeness).

$(\Omega, \mathfrak{M}, \mathcal{H}, E)$ is referred to a spectral measure space.

Basic properties of spectral measures follow from finite additivity.

Theorem 3.4.2 *Let δ_1 and δ_2 be arbitrary measurable subsets of Ω , i. e. $(\delta_1, \delta_2 \in \mathfrak{M})$. Then*

- i) $E(\delta_1)E(\delta_2) = E(\delta_2)E(\delta_1) = E(\delta_1 \cap \delta_2)$ (commutativity).
- ii) $E(\delta_1 \cap \delta_2) = 0$ if $\delta_1 \cap \delta_2 = \emptyset$ (orthogonality).
- iii) $E(\delta_1) \leq E(\delta_2)$ if $\delta_1 \subset \delta_2$ (monotonicity).

With any spectral measure we consider the family of complex measures defined on $\mathfrak{M}$ with $\mu_{f,g}(\delta) \stackrel{\text{def}}{=} \langle E(\delta)f, g \rangle$ for $\delta \in \mathfrak{M}$. A spectral measure generates a family of finite scalar measures on $\mathfrak{M}$. Namely, for $f = g$ we denote the measure $\mu_{f,g}$ simply as μ_f and since $\mu_f(\delta) = \langle E(\delta)f, f \rangle = \|E(\delta)f\|^2 \geq 0$ μ_f is a positive measure and $\mu_f(\Omega) = \|E(\Omega)f\|^2 = \|f\|^2$.

We have $4\mu_{f,g}(\delta) = \mu_{f+g}(\delta) + \mu_{f-g}(\delta) + i\mu_{f+ig}(\delta) - i\mu_{f-ig}(\delta)$.

$$\begin{aligned} \mu_{g,f}(\delta) &= \langle E(\delta)g, f \rangle = \langle g, E(\delta)f \rangle = \overline{\langle E(\delta)f, g \rangle} = \overline{\mu_{f,g}(\delta)}, \\ |\mu_{f,g}(\delta)|^2 &\leq \mu_f(\delta)\mu_g(\delta). \end{aligned}$$

The last inequality follows from

$$|\langle E(\delta)f, g \rangle| = |\langle E(\delta)f, E(\delta)g \rangle| \leq \|E(\delta)f\| \|E(\delta)g\|.$$

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If $\delta_n \in \mathfrak{M}$ for $n \in \mathbb{N}$ are disjoint sets and $\delta = \bigcup_n \delta_n$

$$\begin{aligned} \sum_n |\mu_{f,g}(\delta_n)| &\leq \sum_n \sqrt{\mu_f(\delta_n)} \sqrt{\mu_g(\delta_n)} \leq \left(\sum_n \mu_f(\delta_n) \right)^{1/2} \left(\sum_n \mu_g(\delta_n) \right)^{1/2} \\ &= \sqrt{\mu_f(\delta)} \sqrt{\mu_g(\delta)}. \end{aligned}$$

The variation $|\mu_{f,g}|$ of $\mu_{f,g}$ satisfies

$$|\mu_{f,g}|(\delta) \leq \sqrt{\mu_f(\delta)} \sqrt{\mu_g(\delta)} \quad (f, g \in \mathcal{H}, \delta \in \mathfrak{M}).$$

In particular, for $\delta = \Omega$ we have $|\mu_{f,g}|(\Omega) \leq \|f\| \|g\|$.

If φ is a real measurable function on Ω then

$$E\text{-sup } \varphi = \{a \in \mathbb{R} : \varphi(y) \leq a \quad E - a.e\}$$

A function φ is called simple if there exists a partition $\delta_1, \dots, \delta_N$ of Ω into disjoint measurable subsets such that φ is constant on each δ_n i.e. $\varphi|_{\delta_n} = c_n$ for $n = 1, \dots, N$. If χ_δ is the characteristic function of the set δ then $\varphi = \sum_{n=1}^N c_n \chi_{\delta_n}$. The set $\Pi(\Omega, \mathfrak{M})$ of all simple functions is a dense subalgebra of $\mathbf{L}^\infty(\Omega, E)$, where by $\mathbf{L}^\infty(\Omega, \mathfrak{M})$ is denoted the set of E -bounded E -measurable functions on Ω . $\mathbf{L}^\infty(\Omega, E)$ is endowed with the norm $\|\varphi\|_E = E\text{-sup}|\varphi|$.

Definition 3.4.3 *The integral of $\varphi \in \Pi(\Omega, \mathfrak{M})$ $\varphi = \sum_{n=1}^N c_n \chi_{\delta_n}$ with respect to E is the operator*

$$\mathcal{I}(\varphi) = \int \varphi dE \stackrel{\text{def}}{=} \sum_{n=1}^N c_n E(\delta_n)$$

The definition is independant of the choice of representation. This follows from the finite additivity of spectral measure. Indeed, let $\varphi = \sum_{n=1}^N c_n \chi_{\delta_n} = \sum_{m=1}^M d_m \chi_{\gamma_m}$. Then $\varphi(x) = c_n = d_m$ for all $x \in \delta_n \cap \gamma_m \neq \emptyset$ and $c_n E(\delta_n \cap \gamma_m) = d_m E(\delta_n \cap \gamma_m)$ for all $1 \leq n \leq N$ and $1 \leq m \leq M$. In case when $\delta_n \cap \gamma_m = \emptyset$

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then $E(\delta_n \cap \gamma_m) = 0$ and the previous equation is valid. It follows that

$$\sum_{n=1}^N c_n E(\delta_n) = \sum_{n=1}^N \sum_{m=1}^M c_n E(\delta_n \cap \gamma_m) = \sum_{m=1}^M \sum_{n=1}^N d_m E(\delta_n \cap \gamma_m) = \sum_{m=1}^M d_m E(\gamma_m).$$

The basic properties of the integral defined on $\Pi(\Omega, \mathfrak{M})$ are listed below.

Proposition 3.4.4 *If $\varphi, \psi \in \Pi(\Omega, \mathfrak{M})$, $\alpha, \beta \in \mathbb{C}$ and $f, g \in \mathcal{H}$ then*

- 1) $\mathcal{I}(\alpha\varphi + \beta\psi) = \alpha\mathcal{I}(\varphi) + \beta\mathcal{I}(\psi)$,
- 2) $\mathcal{I}(\varphi\psi) = \mathcal{I}(\varphi)\mathcal{I}(\psi)$,
- 3) $\mathcal{I}(\varphi)^* = \mathcal{I}(\bar{\varphi})$,
- 4) $\mathcal{I}(1) = I$,
- 5) $\langle \mathcal{I}(\varphi)f, g \rangle = \int \varphi d\mu_{f,g}$,
- 6) $\langle \mathcal{I}(\varphi)f, f \rangle = \int \varphi d\mu_f$,
- 7) $\|\mathcal{I}(\varphi)f\|^2 = \int |\varphi|^2 d\mu_f$,
- 8) $\|\mathcal{I}(\varphi)\| = E - \sup |\varphi| = \|\varphi\|_E$.

Now, the definition of the integral extends to $\mathbf{L}^\infty(\Omega, E)$ by passing to the limit, i.e. for $\varphi \in \mathbf{L}^\infty(\Omega, E)$ we put $\mathcal{I}(\varphi) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \mathcal{I}(\varphi_n)$ where $\{\varphi_n\}_{n \in \mathbb{N}}$ is an arbitrary sequence of simple function such that $\|\varphi_n - \varphi\|_E = 0$. Since in $\mathcal{B}(\mathcal{H})$ the linear operations, the multiplication, the norm and the involution $T \mapsto T^*$ are continuous with respect to the convergence in $\mathcal{B}(\mathcal{H})$, it follows that 4) – 6), 7) 11) hold for arbitrary functions in $\mathbf{L}^\infty(\Omega, E)$. The above properties establish the central result of the theory of integration with respect to a spectral measure.

Theorem 3.4.5 *The mapping $\mathcal{I} : \varphi \mapsto \mathcal{I}(\varphi)$ is an isometric isomorphism of the Banach algebra $\mathbf{L}^\infty(\Omega, E)$ with unit $\mathbf{1}$ and involution $\varphi \mapsto \bar{\varphi}$ onto a commutative subalgebra of $\mathcal{B}(\mathcal{H})$ with unit $\mathbf{I}$ and involution $T \mapsto T^*$.*

Self-adjoint, normal and unitary operators admit representations in the form of integrals taken with respect to suitable spectral measures. Such representations are called spectral resolutions (see [3]).

Theorem 3.4.6 *Let A be a self-adjoint operator on $\mathcal{H}$. Then, there exists a unique spectral measure E defined on the σ -algebra of Borel subsets of $[-\|A\|, \|A\|]$ such that*

$$A = \int_{\sigma(A)} \lambda dE(\lambda).$$

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Theorem 3.4.7 *Let A be an arbitrary unitary operator. Then, there exists a unique spectral measure on the Borel subsets of the unit circle $\mathbb{T}$ such that*

$$A = \int_{\mathbb{T}} \lambda dE(\lambda).$$

Theorem 3.4.8 *For any normal operator A there exists a unique spectral measure E defined on Borel subsets of $\sigma(A)$ such that*

$$A = \int_{\sigma(A)} \lambda dE(\lambda).$$

In other words, the spectral theorem for normal operators says that every normal operator is a spectral integral of independent variable taken with respect to its associated spectral measure. If we denote by j the identical mapping, the spectral theorem actually says that $\mathcal{I}(j) = A$. From the Proposition 3.4.4 4) we have $\mathcal{I}(1) = I$ and from the properties 1) and 2) of the same Proposition 3.4.4 it follows that $\mathcal{I}(j^n) = A^n$ and $\mathcal{I}(p) = p(A)$ for any polynomial p . For every $\lambda \notin \sigma(A)$ the function $\varphi_\lambda(z) = \frac{1}{z-\lambda}$ is continuous and therefore bounded on the compact $\sigma(A)$. From 2) of the Proposition 3.4.4 we obtain

$$(A - \lambda I)\mathcal{I}(\varphi_\lambda) = \int_{\sigma(A)} (z - \lambda)dE \int_{\sigma(A)} \frac{1}{z - \lambda}dE = \int_{\sigma(A)} (z - \lambda)\frac{1}{z - \lambda}dE = I$$

which means that it is in fact $\mathcal{I}(\varphi_\lambda) = (A - \lambda I)^{-1}$. From the Proposition 3.4.4 1) and 2) it follows that for any other rational function r with poles outside the $\sigma(A)$ the operator $r(A)$ can be identified with $\mathcal{I}(r)$. Thus, for a wider class of functions, a (normal) operator variable can be allowed, more precisely:

Definition 3.4.9 *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator. For an arbitrary Borel-measurable bounded function φ on $\sigma(A)$ we define*

$$\varphi(A) \stackrel{\text{def}}{=} \int_{\sigma(A)} \varphi dE.$$

The following proposition gives rules for this type of functional calculus.

Proposition 3.4.10 *Let $\varphi, \psi \in L^\infty(\Omega, E)$, $\alpha, \beta \in \mathbb{C}$ and $f, g \in \mathcal{H}$. Then*

- 1) $(\alpha\varphi + \beta\psi)(A) = \alpha\varphi(A) + \beta\psi(A)$,
- 2) $(\varphi\psi)(A) = \varphi(A)\psi(A)$,
- 3) $\varphi(A)^* = \overline{\varphi}(A)$,
- 4) $1(A) = I$,
- 5) $\langle \varphi(A)f, g \rangle = \int \varphi d\mu_{f,g}$,
- 6) $\langle \varphi(A)f, f \rangle = \int \varphi d\mu_f$,
- 7) $\|\varphi(A)f\|^2 = \int |\varphi|^2 d\mu_f$,
- 8) $\|\varphi(A)\| = E - \sup |\varphi| = \|\varphi\|_E$.

3.5 Cauchy-Schwarz norm inequalities in norm ideals of compact operators

We present Cauchy-Schwarz norm inequalities needed to derive inequalities for operator monotone functions and hyponormal operators in the next chapter.

If family $\{\mathcal{C}_t\}_{t \in \Omega}$ in $\mathcal{B}(\mathcal{H})$ consists of mutually commuting normal operators, i. e. $\mathcal{C}_s^* \mathcal{C}_t = \mathcal{C}_t \mathcal{C}_s^*$ for all $t, s \in \Omega$ we will refer to it as a m. c. n. o. family.

Furthermore, if one of the families $\{\mathcal{A}_t\}_{t \in \Omega}$ and $\{\mathcal{B}_t\}_{t \in \Omega}$ consists of mutually commuting normal operators, then we have appropriate Cauchy-Schwarz inequalities for ideals of compact operators $\mathcal{C}_{\Phi(p)}(\mathcal{H})$ and $\mathcal{C}_{\Phi(p)^*}(\mathcal{H})$ when $p \geq 2$.

The following result was proved in [20].

Theorem 3.5.1 *Let $p \geq 2$, $X \in \mathcal{B}(\mathcal{H})$ and Φ be an arbitrary s. n. function. Let $A, B \in L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$. If $\{\mathcal{A}_t\}_{t \in \Omega}$ is a m. c. n. o. family and $X \in \mathcal{C}_{\Phi(p)}(\mathcal{H})$, then*

$$\left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right\|_{\Phi(p)} \leq \left\| \left(\int_{\Omega} \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right)^{1/2} X \right\|_{\Phi(p)} \left\| \int_{\Omega} \mathcal{B}_t^* \mathcal{B}_t d\mu(t) \right\|_{\Phi(p)}^{1/2}. \quad (3.6)$$

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Alternatively, if $A^*, B^* \in L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$, $\{\mathcal{B}_t\}_{t \in \Omega}$ is a m. c. n. o. family and $X \in \mathfrak{C}_{\Phi(p)}(\mathcal{H})$, then

$$\left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right\|_{\Phi(p)} \leq \left\| \int_{\Omega} \mathcal{A}_t \mathcal{A}_t^* d\mu(t) \right\|^{1/2} \left\| X \left(\int_{\Omega} \mathcal{B}_t \mathcal{B}_t^* d\mu(t) \right)^{1/2} \right\|_{\Phi(p)}. \quad (3.7)$$

Proof: For the proof see [20, Theorem 3.1(a)]. ■

Remark 3.5.2 The inequality (3.6) in Theorem 3.5.1 is exactly the inequality (33) in [24, Lemma 3.4] in special case $\mathcal{C}_t = \mathcal{A}_t^*$ and $\mathcal{D}_t = \mathcal{B}_t$ for all $t \in \Omega$, while the inequality (3.7) is exactly the inequality (34) in [24, Lemma 3.4] in special case $\mathcal{C}_t = \mathcal{A}_t^*$ and $\mathcal{D}_t = \mathcal{B}_t$ for all $t \in \Omega$.

In [20, Theorem 3.1 b)] it was shown that the commutativity and normality for any of families $\{\mathcal{A}_t\}_{t \in \Omega}$ and $\{\mathcal{B}_t\}_{t \in \Omega}$ is not required for the validity of (3.6) if $A, B \in L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ and for (3.7) if $A^*, B^* \in L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ if $\|\cdot\|_{\Phi(p)}$ is a Hilbert-Schmidt norm $\|\cdot\|_2$, i.e. the following inequalities hold.

Theorem 3.5.3 Let $A, B \in L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$, $X \in \mathcal{B}(\mathcal{H})$. If $X \in \mathfrak{C}_2(\mathcal{H})$ then

$$\left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right\|_2 \leq \left\| \left(\int_{\Omega} \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right)^{1/2} X \right\|_2 \left\| \int_{\Omega} \mathcal{B}_t^* \mathcal{B}_t d\mu(t) \right\|_2^{1/2}. \quad (3.8)$$

while if $A^*, B^* \in L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ and $X \in \mathfrak{C}_2(\mathcal{H})$

$$\left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right\|_2 \leq \left\| \int_{\Omega} \mathcal{A}_t \mathcal{A}_t^* d\mu(t) \right\|^{1/2} \left\| X \left(\int_{\Omega} \mathcal{B}_t \mathcal{B}_t^* d\mu(t) \right)^{1/2} \right\|_2. \quad (3.9)$$

Proof: For the proof see [20, Theorem 3.1 b)]. ■

The following result was proved in [20] and points out Cauchy-Schwarz norm inequality for the nuclear norm.

Theorem 3.5.4 If $A, B^* \in L_G^2(\Omega, \mu, \mathcal{B}(\mathcal{H}))$ and $X \in \mathfrak{C}_1(\mathcal{H})$ then

$$\left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right\|_1 \leq \left\| \left(\int_{\Omega} \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right)^{1/2} X \left(\int_{\Omega} \mathcal{B}_t \mathcal{B}_t^* d\mu(t) \right)^{1/2} \right\|_1. \quad (3.10)$$

The next theorem considers ideals $\mathfrak{C}_{\Phi(p)^*}(\mathcal{H})$ and was established in [20].

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Theorem 3.5.5 *Let $p \geq 2$, Φ be an arbitrary s. n. function and at least one of families $\{\mathcal{A}_t\}_{t \in \Omega}$ and $\{\mathcal{B}_t\}_{t \in \Omega}$ is a m. c. n. o. family. If $\int_{\Omega} \|\mathcal{A}_t f\|^2 + \|\mathcal{B}_t^* f\|^2 d\mu(t) < +\infty$ for all $f \in \mathcal{H}$, then for all $X \in \mathbf{C}_{\Phi(p)^*}(\mathcal{H})$*

$$\left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right\|_{\Phi(p)^*} \leq \left\| \left(\int_{\Omega} \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right)^{1/2} X \left(\int_{\Omega} \mathcal{B}_t \mathcal{B}_t^* d\mu(t) \right)^{1/2} \right\|_{\Phi(p)^*}. \quad (3.11)$$

Proof: For the proof see [20, Theorem 3.1 d)]. ■

The following result was established in [18].

Theorem 3.5.6 *Let Φ be s. n. function and let $\{\mathcal{A}_t\}_{t \in \Omega}$ and $\{\mathcal{B}_t\}_{t \in \Omega}$ be weakly*-measurable m. c. n. o. families such that $\int_{\Omega} \|\mathcal{A}_t f\|^2 + \|\mathcal{B}_t f\|^2 d\mu(t) < +\infty$ for all $f \in \mathcal{H}$. Then for $X \in \mathbf{C}_{\Phi}(\mathcal{H})$*

$$\left\| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right\|_{\Phi} \leq \left\| \left(\int_{\Omega} \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right)^{1/2} X \left(\int_{\Omega} \mathcal{B}_t \mathcal{B}_t^* d\mu(t) \right)^{1/2} \right\|_{\Phi}. \quad (3.12)$$

Proof: For the proof see [18, Theorem 3.2]. ■

We present now different types of operator Cauchy-Schwarz inequalities which are established in [20].

Theorem 3.5.7 *Let $A^*, B \in L_G^2(\Omega, \mu, \mathbf{B}(\mathcal{H}))$ and $X \in \mathbf{B}(\mathcal{H})$.*

a1) *Then $t \mapsto \mathcal{A}_t X \mathcal{B}_t$ acting on Ω is weak*-integrable and*

$$\left| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right|^2 \leq \left\| \int_{\Omega} \mathcal{A}_t \mathcal{A}_t^* d\mu(t) \right\| \int_{\Omega} \mathcal{B}_t^* X^* X \mathcal{B}_t d\mu(t). \quad (3.13)$$

a2) *For every $\varepsilon > 0$*

$$\left| \left(\varepsilon I + \int_{\Omega} \mathcal{A}_t \mathcal{A}_t^* d\mu(t) \right)^{-1/2} \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right|^2 \leq \int_{\Omega} \mathcal{B}_t^* X^* X \mathcal{B}_t d\mu(t). \quad (3.14)$$

a3) *If $\int_{\Omega} \mathcal{A}_t \mathcal{A}_t^* d\mu(t)$ is additionally invertible, then εI could be omitted in the inequality (3.14).*

a4) *If, in addition $\{\mathcal{A}_t\}_{t \in \Omega}$ is a m. c. n. o. family, then*

$$\left| \int_{\Omega} \mathcal{A}_t X \mathcal{B}_t d\mu(t) \right|^2 \leq \int_{\Omega} \mathcal{B}_t^* X^* \left(\int_{\Omega} \mathcal{A}_t \mathcal{A}_t^* d\mu(t) \right) X \mathcal{B}_t d\mu(t). \quad (3.15)$$

Proof: Inequalities (3.13), (3.14) and (a3) are proved in [20, Lemma 2.1], while the inequality (3.15) is proved in [20, Corollary 2.3]. ■

3.6 Operator monotone functions and hyponormal operators

The following results present new norm inequalities for operator monotone functions, as well as to complement those presented in [18] and [23]. They were proved in [4].

According to [16, Theorem 2.7.7], an operator increasingly monotone function φ on $(0, +\infty)$ admits its unique extension to the Pick class function, also denoted by φ , which is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and satisfies $\Im \varphi(z) > 0$ for all $\Im z > 0$. Moreover, $\varphi(z) = \varphi(0) + bz + \int_0^\infty \frac{zt}{z+t} d\mu(t)$, where $\int_0^\infty \frac{t}{1+t} d\mu(t) < +\infty$ for all z in the open right half plane $\mathbb{H}^+ \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \frac{z+\bar{z}}{2} > 0\}$. Thus $\varphi(A) = bA + \int_0^\infty tA(tI + A)^{-1} d\mu(t)$ for an operator monotone function φ on $[0, +\infty)$ satisfying $\varphi(0) = 0$ and for all strictly accretive operator $A \in \mathcal{B}(\mathcal{H})$.

Lemma 3.6.1 *Let φ be an operator monotone function on $[0, +\infty)$, given with the integral representation (3.5), satisfying $\varphi(0) = 0$. Then $\varphi'(x) = b + \int_0^\infty \frac{t^2}{(x+t)^2} d\mu(t)$ for all $x \in (0, +\infty)$.*

Proof: We start from the integral representation of an operator monotone function φ on $[0, +\infty)$ given by (3.5). Since $\varphi(0) = 0$, it follows that $a = 0$ and $\frac{\varphi(x) - \varphi(x_0)}{x - x_0} = b + \int_0^\infty \frac{t^2}{(x+t)(x_0+t)} d\mu(t)$ for any given $x_0 \in (0, +\infty)$. Therefore,

$$\begin{aligned} & \left| \frac{\varphi(x) - \varphi(x_0)}{x - x_0} - \left(b + \int_0^\infty \frac{t^2}{(x_0 + t)^2} d\mu(t) \right) \right| = \left| \int_0^\infty \frac{t^2(x_0 - x)}{(x_0 + t)^2(x + t)} d\mu(t) \right| \\ & \leq |x - x_0| \int_0^\infty \frac{t^2}{|x_0 + t|^2 |x + t|} d\mu(t) \leq |x - x_0| \int_0^\infty \frac{t}{1+t} \frac{t(t+1)}{|x_0 + t|^2 (t + \frac{x_0}{2})} d\mu(t) \\ & \leq |x - x_0| \int_0^\infty \left(1 + \left| 1 - \frac{1}{x_0} \right| \right) \frac{2}{x_0} \frac{t}{1+t} d\mu(t) < \varepsilon \end{aligned} \quad (3.16)$$

for $|x - x_0| < \frac{x_0}{2}$ and $|x - x_0| < \frac{\varepsilon}{C \int_0^\infty \frac{t}{1+t} d\mu(t)}$, where $C := C_{x_0} := \left(1 + \left| 1 - \frac{1}{x_0} \right| \right) \frac{2}{x_0}$ is a constant and $0 < \int_0^\infty \frac{t}{1+t} d\mu(t) < +\infty$. The second inequality in (3.16) follows because $|x - x_0| < \frac{x_0}{2}$ implies $\frac{x_0}{2} < x < \frac{3x_0}{2}$ and $\frac{1}{x+t} < \frac{1}{t + \frac{x_0}{2}}$. We obtain the last inequality in (3.16) by the following estimates $\frac{t+1}{|x_0+t|} = \left| 1 + \frac{1-x_0}{x_0+t} \right| \leq 1 + \frac{|1-x_0|}{x_0+t} \leq 1 + \frac{|1-x_0|}{x_0} = 1 + \left| 1 - \frac{1}{x_0} \right|$, $\frac{1}{t + \frac{x_0}{2}} \leq \frac{2}{x_0}$ and $\frac{t}{x_0+t} \leq 1$. The case when $\int_0^\infty \frac{t}{1+t} d\mu(t) = 0$ is trivial. ■

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Lemma 3.6.2 *If $\varphi : [0, +\infty) \mapsto \mathbb{R}$ is operator monotone function satisfying $\varphi(0) = 0$ and operators $A, B, X \in \mathfrak{B}(\mathcal{H})$ are such that A and B are accretive, then*

$$AX\varphi(B) - \varphi(A)XB = \int_0^\infty t(tI + A)^{-1}A(AX - XB)B(tI + B)^{-1}d\mu(t). \quad (3.17)$$

Moreover, if A is also cohyponormal, then

$$(tI + A^*)^{-1}(tI + A)^{-1} \leq (tI + \frac{A+A^*}{2})^{-2}. \quad (3.18)$$

Proof: The condition $\varphi(0) = 0$ is equivalent to the fact that $a = 0$ in the formula (3.5), so $\varphi(x) = bx + \int_0^\infty \frac{tx}{x+t}d\mu(t)$ for all x satisfying $x \in [0, +\infty)$, where $\int_0^\infty \frac{t}{t+1}d\mu(t) < +\infty$. This implies

$$\begin{aligned} AX\varphi(B) - \varphi(A)XB &= AX\left(bB + \int_0^\infty tB(tI + B)^{-1}d\mu(t)\right) \\ &- \left(bA + \int_0^\infty tA(tI + A)^{-1}d\mu(t)\right)XB \\ &= \int_0^\infty t(AXB(tI + B)^{-1} - A(tI + A)^{-1}XB)d\mu(t) \\ &= \int_0^\infty t(tI + A)^{-1}A(AX - XB)B(tI + B)^{-1}d\mu(t). \end{aligned} \quad (3.19)$$

The last equality in (3.19) follows from the following calculus

$$\begin{aligned} &AXB(tI + B)^{-1} - A(tI + A)^{-1}XB \\ &= (tI + A)^{-1}((tI + A)AXB - AXB(tI + B))(tI + B)^{-1} \\ &= (tI + A)^{-1}(A^2XB - AXB^2)(tI + B)^{-1} \\ &= (tI + A)^{-1}A(AX - XB)B(tI + B)^{-1} \end{aligned}$$

If A is additionally cohyponormal, then

$$\begin{aligned} (tI + A)(tI + A^*) &= t^2I + t(A + A^*) + AA^* \\ &\geq t^2I + 2t\frac{A+A^*}{2} + \frac{AA^*+A^*A}{2} = t^2I + 2t\frac{A+A^*}{2} + \left(\frac{A+A^*}{2}\right)^2 + \left(\frac{A-A^*}{2i}\right)^2 \\ &= \left(tI + \frac{A+A^*}{2}\right)^2 + \left(\frac{A-A^*}{2i}\right)^2 \geq \left(tI + \frac{A+A^*}{2}\right)^2, \end{aligned}$$

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which proves the inequality (3.18) since the mapping $(0, \infty) \rightarrow (0, \infty): t \mapsto t^{-1}$ is operator monotone decreasing, due to [16, Lemma 2.5.5]. ■

The previous lemma 3.6.2 implies

Lemma 3.6.3 *Let $A \in \mathfrak{B}(\mathcal{H})$ be a strictly accretive operator, satisfying $A_{\Re} \geq cI$ for some $c > 0$ and let φ be an operator monotone function on $[0, +\infty)$, such that $\varphi(0) = 0$. Then $\varphi(c) - c\varphi'(c) > 0$ and*

$$\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right) \geq (\varphi(c) - c\varphi'(c))I, \quad (3.20)$$

so that $\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)$ is strictly positively definite and invertible.

Proof: Consider the function $g(x) = \varphi(x) - x\varphi'(x)$ on $[0, +\infty)$. Since $g'(x) \geq 0$ according to the properties of operator monotone functions (see [16, Corollary 2.5.4]), it follows that g is an increasing real function on $[0, +\infty)$. Therefore if $A_{\Re} \geq cI$ for some positive scalar $c > 0$ it follows $g(A_{\Re}) \geq g(c)I > 0$. Indeed from the spectral calculus we have $g(A_{\Re} - cI) = \int_{\sigma(A_{\Re})} g((\lambda) - g(c))dE(\lambda)$ and $\langle (g(A_{\Re}) - cg(I))h, h \rangle = \int_{\sigma(A_{\Re})} (g(t) - g(c))d\mu_h(t)$ is strictly positive where E is the spectral measure associated to $A_{\Re}$ and μ_h is the associated (scalar) measure for an arbitrary $h \in \mathcal{H}$, given by $d\mu_h(\delta) = \langle E(\delta)h, h \rangle$ for every Borel set $\delta \subset \mathbb{R}$. ■

Theorem 3.6.4 *Let Ψ, Φ be s.n. functions, let $p \geq 2$, and let φ be an operator monotone function on $[0, +\infty)$, such that $\varphi(0) = 0$ and let $A, B, X \in \mathfrak{B}(\mathcal{H})$. If A and B are strictly accretive, such that $AX - XB \in \mathfrak{C}_{\Psi}(\mathcal{H})$, then $AX\varphi(B) - \varphi(A)XB \in \mathfrak{C}_{\Psi}(\mathcal{H})$ as well, satisfying*

$$\begin{aligned} \|AX\varphi(B) - \varphi(A)XB\|_{\Psi} &\leq \left\| \sqrt{\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)} \left(\frac{A+A^*}{2}\right)^{-1} \right. \\ &\quad \left. A(AX - XB)B\left(\frac{B+B^*}{2}\right)^{-1} \sqrt{\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)} \right\|_{\Psi} \end{aligned} \quad (3.21)$$

a1) if both A and B are normal,

a2) if A is cohyponormal, B is hyponormal and at least one of them is normal, while $\Psi := \Phi^{(p)*}$,

a3) if A is cohyponormal, B is hyponormal and $\Psi := \ell^1$;

$$\begin{aligned} & \left\| (AX\varphi(B) - \varphi(A)XB)^{\frac{B+B^*}{2}} \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right) \right)^{-1/2} \right\|_{\Psi} \\ & \leq \left\| \sqrt{\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)} \left(\frac{A+A^*}{2}\right)^{-1} A(AX - XB)B \right\|_{\Psi}, \end{aligned} \quad (3.22)$$

b1) if A is normal, B cohyponormal and $\Psi = \Phi^{(p)}$,

b2) if A and B are both cohyponormal and $\|\cdot\|_{\Psi} = \|\cdot\|_2$,

$$\begin{aligned} & \left\| \left(\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right) \right)^{-1/2} \frac{A+A^*}{2} (AX\varphi(B) - \varphi(A)XB) \right\|_{\Psi} \\ & \leq \left\| A(AX - XB)B \left(\frac{B+B^*}{2}\right)^{-1} \sqrt{\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)} \right\|_{\Psi}, \end{aligned} \quad (3.23)$$

c1) if A is hyponormal, B is normal and $\Psi = \Phi^{(p)}$,

c2) if A and B are both hyponormal and $\|\cdot\|_{\Psi} = \|\cdot\|_2$,

$$\begin{aligned} & \left\| \frac{A+A^*}{2} (AX\varphi(B) - \varphi(A)XB)^{\frac{B+B^*}{2}} \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right) \right)^{-1/2} \right\|_{\Psi} \\ & \leq \left\| \sqrt{\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)} A(AX - XB)B \right\|_{\Psi}, \end{aligned} \quad (3.24)$$

d1) if A is normal, B is cohyponormal and $\Psi := \Phi^{(p)}$,

$$\begin{aligned} & \left\| \left(\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right) \right)^{-1/2} \frac{A+A^*}{2} (AX\varphi(B) - \varphi(A)XB)^{\frac{B+B^*}{2}} \right\|_{\Psi} \\ & \leq \left\| A(AX - XB)B \sqrt{\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)} \right\|_{\Psi}, \end{aligned} \quad (3.25)$$

e1) if A is hyponormal, B is normal and $\Psi := \Phi^{(p)}$.

Proof: To prove a1), let us first note that $\{\mathcal{A}_t\}_{t \geq 0}$ given by $\mathcal{A}_t := \sqrt{t}(tI + A)^{-1} \frac{A+A^*}{2}$ is the $[\mu]_{\infty}$ s.i. family, satisfying

$$\int_0^{\infty} \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \leq \varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right). \quad (3.26)$$

Indeed, the estimate (3.26) is based on

$$\begin{aligned} 0 & \leq \int_0^{\infty} \mathcal{A}_t^* \mathcal{A}_t d\mu(t) = \int_0^{\infty} t \frac{A+A^*}{2} ((tI + A)(tI + A^*))^{-1} \frac{A+A^*}{2} d\mu(t) \\ & \leq \int_0^{\infty} t \frac{A+A^*}{2} (tI + \frac{A+A^*}{2})^{-2} \frac{A+A^*}{2} d\mu(t) \\ & = \int_0^{\infty} t (tI + \frac{A+A^*}{2} - tI) (tI + \frac{A+A^*}{2})^{-2} \frac{A+A^*}{2} d\mu(t) \\ & = \int_0^{\infty} t \frac{A+A^*}{2} (tI + \frac{A+A^*}{2})^{-1} d\mu(t) - \frac{A+A^*}{2} \int_0^{\infty} t^2 (tI + \frac{A+A^*}{2})^{-2} d\mu(t) \\ & = \varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right). \end{aligned} \quad (3.27)$$

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The second inequality in (3.27) follows from (3.18) since A is normal and therefore cohyponormal as well. By analogy, denoting $\{\mathcal{B}_t\}_{t \geq 0}$ the family given by $\mathcal{B}_t := \sqrt{t} \frac{B+B^*}{2} (tI + B)^{-1}$, we see that $\{\mathcal{B}_t^*\}_{t \geq 0}$ is another $[\mu]$ s.i. family, which satisfies

$$0 \leq \int_0^\infty \mathcal{B}_t \mathcal{B}_t^* d\mu(t) \leq \varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2} \varphi'\left(\frac{B+B^*}{2}\right). \quad (3.28)$$

Starting from the formula (3.17), an application of Cauchy-Schwarz norm inequality (3.12) in Theorem 3.5.6 to $[\mu]$ s.i. families $\{\mathcal{A}_t\}_{t \geq 0}$, $\{\mathcal{B}_t^*\}_{t \geq 0}$ and $Y := \left(\frac{A+A^*}{2}\right)^{-1} A(AX - XB)B \left(\frac{B+B^*}{2}\right)^{-1}$ (instead of X) we get the next inequality in (3.29),

$$\begin{aligned} \|AX\varphi(B) - \varphi(A)XB\|_\Psi &= \left\| \int_0^\infty \mathcal{A}_t Y \mathcal{B}_t d\mu(t) \right\|_\Psi \\ &\leq \left\| \left(\int_0^\infty \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right)^{1/2} Y \left(\int_0^\infty \mathcal{B}_t \mathcal{B}_t^* d\mu(t) \right)^{1/2} \right\|_\Psi \end{aligned} \quad (3.29)$$

$$\begin{aligned} &\leq \left\| \sqrt{\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2} \varphi'\left(\frac{A+A^*}{2}\right)} \left(\frac{A+A^*}{2}\right)^{-1} A(AX - XB)B \right. \\ &\quad \left. \left(\frac{B+B^*}{2}\right)^{-1} \sqrt{\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2} \varphi'\left(\frac{B+B^*}{2}\right)} \right\|_\Psi, \end{aligned} \quad (3.30)$$

while the double monotonicity property (3.1), combined by (3.26) and (3.28) justifies the inequality in (3.30).

To prove the inequality (3.21) in the case a2), we apply Cauchy-Schwarz norm inequality (3.11) in Theorem 3.5.5 instead of (3.12) in Theorem 3.5.6 to the same families $\{\mathcal{A}_t\}_{t \geq 0}$ and $\{\mathcal{B}_t\}_{t \geq 0}$ and for the same Y appearing in the proof of a1).

The case a3) of the formula (3.21) is proved by analogy to the proofs already given, but this time by applying Cauchy-Schwarz norm inequality 3.10 for the trace ideals in Theorem 3.5.4. to the same families $\{\mathcal{A}_t\}_{t \geq 0}$ and $\{\mathcal{B}_t\}_{t \geq 0}$ and Y as above.

b1) To prove the inequality (3.22) in the case b1), let us note that $\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2} \varphi'\left(\frac{B+B^*}{2}\right)$ according to Lemma 3.6.3. It follows that

$$\begin{aligned} &(AX\varphi(B) - \varphi(A)XB) \frac{B+B^*}{2} \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2} \varphi'\left(\frac{B+B^*}{2}\right) \right)^{-1/2} \\ &= \left(\int_0^\infty t(tI + A)^{-1} A(AX - XB)B(tI + B)^{-1} d\mu(t) \right) \frac{B+B^*}{2} \\ &\quad \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2} \varphi'\left(\frac{B+B^*}{2}\right) \right)^{-1/2} = \int_0^\infty \mathcal{A}_t Y \mathcal{B}_t d\mu(t), \end{aligned}$$

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$\mathcal{A}_t := \sqrt{t}(tI + A)^{-1} \frac{A+A^*}{2}$, $\mathcal{B}_t := \sqrt{t}(tI + B)^{-1} \frac{B+B^*}{2} \left(\varphi \left(\frac{B+B^*}{2} \right) - \frac{B+B^*}{2} \varphi' \left(\frac{B+B^*}{2} \right) \right)^{-\frac{1}{2}}$ and $Y := \left(\frac{A+A^*}{2} \right)^{-1} A(AX - XB)B$. By applying Cauchy-Schwarz inequality version for Q -norms (3.6) in Theorem 3.5.1 it follows that

$$\begin{aligned} \left\| \int_0^\infty \mathcal{A}_t Y \mathcal{B}_t d\mu(t) \right\|_{\Phi(p)} &\leq \left\| \left(\int_0^\infty \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right)^{1/2} Y \right\|_{\Phi(p)} \left\| \int_0^\infty \mathcal{B}_t^* \mathcal{B}_t d\mu(t) \right\|_{\Phi(p)}^{1/2} = \\ &\left\| \sqrt{\int_0^\infty t \frac{A+A^*}{2} (tI + A^*)^{-1} (tI + A)^{-1} \frac{A+A^*}{2} d\mu(t) \left(\frac{A+A^*}{2} \right)^{-1} A(AX - XB)B} \right\|_{\Phi(p)} \times \\ &\left\| \left(\varphi \left(\frac{B+B^*}{2} \right) - \frac{B+B^*}{2} \varphi' \left(\frac{B+B^*}{2} \right) \right)^{-1/2} \int_0^\infty t \frac{B+B^*}{2} (tI + B^*)^{-1} (tI + B)^{-1} \frac{B+B^*}{2} d\mu(t) \right. \\ &\left. \left(\varphi \left(\frac{B+B^*}{2} \right) - \frac{B+B^*}{2} \varphi' \left(\frac{B+B^*}{2} \right) \right)^{-1/2} \right\|_{\Phi(p)}^{1/2} \\ &\leq \left\| \sqrt{\varphi \left(\frac{A+A^*}{2} \right) - \frac{A+A^*}{2} \varphi' \left(\frac{A+A^*}{2} \right)} \left(\frac{A+A^*}{2} \right)^{-1} A(AX - XB)B \right\|_{\Phi(p)}. \end{aligned} \quad (3.31)$$

The inequality (3.31) is obtained analogously as in Theorem 3.6.4 a) according to the normality of operator A and the cohyponormality of B , combined with the fact that $(0, +\infty) \rightarrow (0, +\infty): t \mapsto t^{-1}$ is operator monotone decreasing function, as well as the double monotonicity property (3.1) for u.i. norms.

b2) For the proof of the inequality (3.22) in this case we replace $\|\cdot\|_{\Phi(p)}$ by $\|\cdot\|_2$ and apply Cauchy-Schwarz norm inequality (3.8) in Theorem 3.5.3 on the same families $\mathcal{A}_t$ and $\mathcal{B}_t$ and the same Y instead of Cauchy-Schwarz inequality (3.6) in Theorem 3.5.1 on those families and Y .

c1) In this case the inequality (3.23) proves in a similar way as the inequality (3.22) by applying Lemma 3.6.3 and Cauchy-Schwarz norm inequality (3.7) in Theorem 3.5.1 on the families

$$\mathcal{A}_t := \sqrt{t} \left(\varphi \left(\frac{A+A^*}{2} \right) - \frac{A+A^*}{2} \varphi' \left(\frac{A+A^*}{2} \right) \right)^{-1/2} \frac{A+A^*}{2} (tI + A)^{-1}, \quad (3.32)$$

$\mathcal{B}_t := \sqrt{t} \frac{B+B^*}{2} (tI + B)^{-1}$, where $\mathcal{B}_t$ consists of commuting normal operators, operator $Y := A(AX - XB)B \left(\frac{B+B^*}{2} \right)^{-1}$ and by using the hypomormality for the operator A .

c2) Similarly, the proof of the inequality (3.23) in this case requires to replace $\|\cdot\|_{\Phi(p)}$ by $\|\cdot\|_2$ and to apply the Cauchy-Schwarz norm inequality

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(3.9) in Theorem 3.5.3 instead of Cauchy-Schwarz norm inequality (3.7) in Theorem 3.5.1 to the same families $\mathcal{A}_t$ and $\mathcal{B}_t$ and the same Y .

d1) To prove the inequality (3.24) in this case, we note that $\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)$ is invertible due to Lemma 3.6.3. It follows that

$$\begin{aligned} & \frac{A+A^*}{2}(AX\varphi(B) - \varphi(A)XB)\frac{B+B^*}{2}\left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)\right)^{-1/2} \\ &= \frac{A+A^*}{2}\left(\int_0^\infty t(tI+A)^{-1}A(AX-XB)B(tI+B)^{-1}d\mu(t)\right)\frac{B+B^*}{2} \\ & \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)\right)^{-1/2} = \int_0^\infty \mathcal{A}_t Y \mathcal{B}_t d\mu(t), \end{aligned}$$

$\mathcal{A}_t := \sqrt{t}\frac{A+A^*}{2}(tI+A)^{-1}$, $\mathcal{B}_t := \sqrt{t}(tI+B)^{-1}\frac{B+B^*}{2}\left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)\right)^{-\frac{1}{2}}$ and $Y := A(AX-XB)B$. By applying Cauchy-Schwarz inequality version for Q -norms (3.6) in Theorem 3.5.1 it follows that

$$\begin{aligned} & \left\| \int_0^\infty \mathcal{A}_t Y \mathcal{B}_t d\mu(t) \right\|_{\Phi(p)} \leq \left\| \left(\int_0^\infty \mathcal{A}_t^* \mathcal{A}_t d\mu(t) \right)^{1/2} Y \right\|_{\Phi(p)} \left\| \int_0^\infty \mathcal{B}_t^* \mathcal{B}_t d\mu(t) \right\|_{\Phi(p)}^{1/2} \\ &= \left\| \sqrt{\int_0^\infty \frac{A+A^*}{2} t(tI+A)^{-1} (tI+A^*)^{-1} \frac{A+A^*}{2} d\mu(t)} A(AX-XB)B \right\|_{\Phi(p)} \times \\ & \left\| \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right) \right)^{-1/2} \int_0^\infty t \frac{B+B^*}{2} (tI+B^*)^{-1} (tI+B)^{-1} \frac{B+B^*}{2} d\mu(t) \right. \\ & \left. \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right) \right)^{-1/2} \right\|_{\Phi(p)}^{1/2} \\ & \leq \left\| \sqrt{\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)} A(AX-XB)B \right\|_{\Phi(p)}. \end{aligned} \quad (3.33)$$

The inequality (3.33) is obtained similarly as in previous cases using normality of operator A and the cohyponormality of B , combined with the fact that $(0, +\infty) \rightarrow (0, +\infty): t \mapsto t^{-1}$ is an operator monotone decreasing function, and the double monotonicity property (3.1) for u.i. norms.

e1) The inequality in (3.25) proves in a similar way as the inequality (3.24) by applying Lemma 3.6.3 and Cauchy-Schwarz norm inequality (3.7) in Theorem 3.5.1 for the families

$$\mathcal{A}_t := \sqrt{t}\left(\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)\right)^{-1/2} \frac{A+A^*}{2}(tI+A)^{-1}, \quad (3.34)$$

$\mathcal{B}_t := \sqrt{t}(tI+B)^{-1}\frac{B+B^*}{2}$, where $\mathcal{B}_t$ consists of commuting normal operators, operator $Y := A(AX-XB)B$ and by using the hypomormality for the operator A .

Theorem 3.6.5 *Let Ψ be s.n. function, let φ be an operator monotone function on $[0, +\infty)$ such that $\varphi(0) = 0$ and let $A, B \in \mathfrak{B}(\mathcal{H})$. If A and B are strictly accretive normal operators, then for all $X \in \mathfrak{B}(\mathcal{H})$ such that $AX - XB \in \mathfrak{C}_\Psi(\mathcal{H})$, we have $AX\varphi(B) - \varphi(A)XB \in \mathfrak{C}_\Psi(\mathcal{H})$ as well, satisfying*

$$\begin{aligned} & \left\| \left(\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right) \right)^{-1/2} (AX\varphi(B) - \varphi(A)XB) \right. \\ & \left. \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right) \right)^{-1/2} \right\|_\Psi \\ & \leq \left\| \left(\frac{A+A^*}{2} \right)^{-1} A (AX - XB) B \left(\frac{B+B^*}{2} \right)^{-1} \right\|_\Psi, \end{aligned} \quad (3.35)$$

$$\begin{aligned} & \left\| \frac{A+A^*}{2} (AX\varphi(B) - \varphi(A)XB) \frac{B+B^*}{2} \right\|_\Psi \leq \left\| \sqrt{\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)} \right. \\ & \left. A (AX - XB) B \sqrt{\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)} \right\|_\Psi, \end{aligned} \quad (3.36)$$

$$\begin{aligned} & \left\| \frac{A+A^*}{2} \left(\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right) \right)^{-1/2} (AX\varphi(B) - \varphi(A)XB) \right. \\ & \left. \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right) \right)^{-1/2} \frac{B+B^*}{2} \right\|_\Psi \leq \|A(AX - XB)B\|_\Psi. \end{aligned} \quad (3.37)$$

Proof: The proof goes by analogy to the proof of Theorem 3.6.4a). $\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)$ and $\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)$ are strictly positively definite and invertible according to Lemma 3.6.3. To prove (3.35) we just apply (3.29) to $\left(\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)\right)^{-1/2} Y \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)\right)^{-1/2}$ instead of Y , where Y is the same as in Theorem 3.6.4. Similarly, (3.36) follows by direct application of (3.29) to $\frac{A+A^*}{2} Y \frac{B+B^*}{2}$ instead of Y , and to prove (3.37) we apply (3.29) to the operator $\frac{A+A^*}{2} \left(\varphi\left(\frac{A+A^*}{2}\right) - \frac{A+A^*}{2}\varphi'\left(\frac{A+A^*}{2}\right)\right)^{-1/2} Y \left(\varphi\left(\frac{B+B^*}{2}\right) - \frac{B+B^*}{2}\varphi'\left(\frac{B+B^*}{2}\right)\right)^{-1/2} \frac{B+B^*}{2}$ instead of Y . ■

Similarly as in the proof of Lemma 3.6.2, by applying Lemma 3.6.3 the following can be obtained.

Corollary 3.6.6 *If $A, B \in \mathfrak{B}(\mathcal{H})$, where A is strictly accretive and cohyponormal and B is strictly accretive and hyponormal, and if φ is an operator monotone function on $[0, +\infty)$ such that $\varphi(0) = 0$, then*

$$\begin{aligned}
 & \left(\varphi \left(\frac{A+A^*}{2} \right) - \frac{A+A^*}{2} \varphi' \left(\frac{A+A^*}{2} \right) \right)^{-\frac{1}{2}} \int_0^\infty t^{\frac{A+A^*}{2}} (tI + A^*)^{-1} (tI + A)^{-1} \frac{A+A^*}{2} d\mu(t) \\
 & \left(\varphi \left(\frac{A+A^*}{2} \right) - \frac{A+A^*}{2} \varphi' \left(\frac{A+A^*}{2} \right) \right)^{-\frac{1}{2}} \leq I. \\
 & \left(\varphi \left(\frac{B+B^*}{2} \right) - \frac{B+B^*}{2} \varphi' \left(\frac{B+B^*}{2} \right) \right)^{-\frac{1}{2}} \int_0^\infty t^{\frac{B+B^*}{2}} (tI + B)^{-1} (tI + B^*)^{-1} \frac{B+B^*}{2} d\mu(t) \\
 & \left(\varphi \left(\frac{B+B^*}{2} \right) - \frac{B+B^*}{2} \varphi' \left(\frac{B+B^*}{2} \right) \right)^{-\frac{1}{2}} \leq I.
 \end{aligned}$$

Theorem 3.6.7 *Let Ψ be s.n. function, let $A, B \in \mathfrak{B}(\mathcal{H})$ be strictly accretive normal operators, $X \in \mathfrak{B}(\mathcal{H})$ and $\theta \in (0, 1)$. If $AX - XB \in \mathfrak{C}_\Psi(\mathcal{H})$ then $AX \log(I + B) - \log(I + A)XB$, $AXB^\theta - A^\theta XB \in \mathfrak{C}_\Psi(\mathcal{H})$ as well and*

$$\begin{aligned}
 & \left\| \frac{A+A^*}{2} (AX \log(I + B) - \log(I + A)XB) \frac{B+B^*}{2} \right\|_\Psi \\
 & \leq \left\| \sqrt{\log \left(I + \frac{A+A^*}{2} \right) - \frac{A+A^*}{2} \left(I + \frac{A+A^*}{2} \right)^{-1}} A(AX - XB)B \right. \\
 & \left. \sqrt{\log \left(I + \frac{B+B^*}{2} \right) - \frac{B+B^*}{2} \left(I + \frac{B+B^*}{2} \right)^{-1}} \right\|_\Psi, \tag{3.38}
 \end{aligned}$$

$$\left\| AXB^\theta - A^\theta XB \right\|_\Psi \leq (1 - \theta) \left\| \left(\frac{A+A^*}{2} \right)^{\frac{\theta}{2}-1} A(AX - XB)B \left(\frac{B+B^*}{2} \right)^{\frac{\theta}{2}-1} \right\|_\Psi, \tag{3.39}$$

$$\begin{aligned}
 & \left\| \left(\frac{A+A^*}{2} \right)^{-\frac{\theta}{2}} (AXB^\theta - A^\theta XB) \left(\frac{B+B^*}{2} \right)^{-\frac{\theta}{2}} \right\|_\Psi \\
 & \leq (1 - \theta) \left\| \left(\frac{A+A^*}{2} \right)^{-1} A(AX - XB)B \left(\frac{B+B^*}{2} \right)^{-1} \right\|_\Psi, \tag{3.40}
 \end{aligned}$$

$$\begin{aligned}
 & \left\| \frac{A+A^*}{2} (AXB^\theta - A^\theta XB) \frac{B+B^*}{2} \right\|_\Psi \\
 & \leq (1 - \theta) \left\| \left(\frac{A+A^*}{2} \right)^{\frac{\theta}{2}} A(AX - XB)B \left(\frac{B+B^*}{2} \right)^{\frac{\theta}{2}} \right\|_\Psi, \tag{3.41}
 \end{aligned}$$

$$\left\| \left(\frac{A+A^*}{2} \right)^{1-\frac{\theta}{2}} (AXB^\theta - A^\theta XB) \left(\frac{B+B^*}{2} \right)^{1-\frac{\theta}{2}} \right\|_\Psi \leq (1 - \theta) \|A(AX - XB)B\|_\Psi. \tag{3.42}$$

Proof: The application of the inequality (3.36) on the operator monotone function $\varphi(t) := \log(1 + t)$ gives (3.38), while the applicatin of the inequalities (3.21), (3.35), (3.36) and (3.37) on the operator monotone function $\varphi(t) := t^\theta$ for $\theta \in (0, 1)$ gives (3.39), (3.40), (3.41) and (3.42), respectively.

■

CHAPTER 3. RELATIONS OF SCHUR-MULTIPLIERS TO
DERIVATION'S INEQUALITIES OF OPERATORS

Theorem 3.6.8 *Let $A, B \in \mathfrak{B}(\mathcal{H})$ be strictly accretive operators, $X \in \mathfrak{B}(\mathcal{H})$ such that $AX - XB \in \mathfrak{C}_{\Phi(p)}(\mathcal{H})$, $p \geq 2$ and let $\theta \in (0, 1)$. Then $AX \log(I + B) - \log(I + A)XB$, $AXB^\theta - A^\theta XB \in \mathfrak{C}_\Psi(\mathcal{H})$ as well and*

a) *If A is hyponormal and B is normal then*

$$\begin{aligned} & \left\| \left(\log \left(I + \frac{A+A^*}{2} \right) - \frac{A+A^*}{2} \left(I + \frac{A+A^*}{2} \right)^{-1} \right)^{-1/2} \frac{A+A^*}{2} (AX \log(I + B) \right. \\ & \left. - \log(I + A)XB) \frac{B+B^*}{2} \right\|_{\Phi(p)} \\ & \leq \left\| A(AX - XB)B \sqrt{\log \left(I + \frac{B+B^*}{2} \right) - \frac{B+B^*}{2} \left(I + \frac{B+B^*}{2} \right)^{-1}} \right\|_{\Phi(p)}, \end{aligned} \quad (3.43)$$

$$\begin{aligned} & \left\| \left(\frac{A+A^*}{2} \right)^{1-\frac{\theta}{2}} (AXB^\theta - A^\theta XB) \frac{B+B^*}{2} \right\|_{\Phi(p)} \\ & \leq (1 - \theta) \left\| A(AX - XB)B \left(\frac{B+B^*}{2} \right)^{\frac{\theta}{2}} \right\|_{\Phi(p)}. \end{aligned} \quad (3.44)$$

b) *If A is normal and B cohyponormal then*

$$\begin{aligned} & \left\| \frac{A+A^*}{2} (AX \log(I + B) - \log(I + A)XB) \frac{B+B^*}{2} (\log \left(I + \frac{B+B^*}{2} \right) \right. \\ & \left. - \frac{B+B^*}{2} \left(I + \frac{B+B^*}{2} \right)^{-1} \right)^{-1/2} \right\|_{\Phi(p)} \\ & \leq \left\| \sqrt{\log \left(I + \frac{A+A^*}{2} \right) - \frac{A+A^*}{2} \left(I + \frac{A+A^*}{2} \right)^{-1}} A(AX - XB)B \right\|_{\Phi(p)}, \end{aligned} \quad (3.45)$$

$$\begin{aligned} & \left\| \frac{A+A^*}{2} (AXB^\theta - A^\theta XB) \left(\frac{B+B^*}{2} \right)^{1-\frac{\theta}{2}} \right\|_{\Phi(p)} \\ & \leq (1 - \theta) \left\| \left(\frac{A+A^*}{2} \right)^{\frac{\theta}{2}} A(AX - XB)B \right\|_{\Phi(p)}. \end{aligned} \quad (3.46)$$

Proof: The inequalities (3.43) and (3.45) are direct consequences of the application of inequalities (3.25) and (3.24) to the operator monotone function $[0, +\infty) \rightarrow [0, +\infty): t \mapsto \log(1 + t)$, while the inequalities (3.44) and (3.46) are proved by applying the inequalities (3.25) and (3.24) to the operator monotone function $[0, +\infty) \rightarrow [0, +\infty): t \mapsto t^\theta$ for $0 < \theta < 1$. ■

Chapter 4

Bibliography

References

- [1] C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press, Inc., Orlando, 1988.
- [2] R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics **169**, Springer-Verlag, New York, 1997.
- [3] M. Sh. Birman, M. Z. Solomyak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, D. Reidel Publishing Company, Dordrecht, 1987.
- [4] K. Bogdanović, *A class of norm inequalities for operator monotone functions and hyponormal moperators*, Complex Anal. Oper. Theory **12** (2024), 18–32.
- [5] K. Bogdanović, A. Peperko, *Hadamard weighted geometric mean inequalities for the spectral and essential spectral radius of positive kernel operators on Banach function and sequence spaces*, Positivity **26:25** (2022).
- [6] K. Bogdanović, A. Peperko, *Inequalities and equalities on the joint and generalized spectral ans essential spectral radius of the Hadamard geometric mean of bounded sets of positive kernel operators*, Linear Mult. Algebra **71**, 2839–2857.

- [7] K. Bogdanović, A. Peperko, *Monotonicity properties of weighted geometric symmetrizations*, Journal of Mathematical Inequalities **18** (2024), 1535–1546.
- [8] J. Diestel, J. J. Uhl, *Vector measures*, Math Surveys Monographs, Vol. 15, Amer. Math. Soc., Providence, RI, 1977, MR56:12216.
- [9] R. Drnovšek, *Sequences of bounds for the spectral radius of a positive operators*, Linear Algebra Appl. **574** (2019), 40–45.
- [10] R. Drnovšek, A. Peperko, *Inequalities for the Hadamard weighted geometric mean of positive kernel operators on Banach function spaces*, Positivity **10** (2006), 613–626.
- [11] R. Drnovšek, A. Peperko, *On the spectral radius of positive operators on Banach sequence spaces*, Linear Algebra Appl. **433** (2010), 241–247.
- [12] P. A. Fillmore, J. G. Stampfli, J. P. Williams, *On the essential numerical range, the essential spectrum, and a problem of Halmos*, Acta Sci. Math. (Szeged) **33**, (1972), 179–192.
- [13] I. Gohberg, S. Goldberg and M. A. Kaashoek, *Classes of linear operators*, Operator Theory Vol. **49**, Birkhäuser Verlag, Basel 1990.
- [14] I. C. Gohberg, M. G. Krein, *Introduction to the Theory of Linear Non-selfadjoint Operators*, Transl. Math. Monographs, **18**, Amer. Math. Soc. Providence, RI, 1969.
- [15] M. Haase, *Convexity inequalities for positive operators*, Positivity **11** (2007), 57–68.
- [16] F. Hiai, *Matrix Analysis: Matrix Monotone Functions, Matrix Means, and Majorization*, Inter. Inf. Sci. Vol. 16, **2** (2010), 139–248.
- [17] F. Hiai, H. Kosaki, *Means of Hilbert Space Operators*, Springer, 2003.
- [18] D. R. Jocić, *Cauchy-Schwarz norm inequalities for weak*-integrals of operator valued functions*, Journal of Functional Analysis **218** (2005), 318–346.

- [19] D. R. Jocić, *Multipliers of elementary operators and comparison of row and column space Schatten p norms*, Linear Algebra Appl. **431** (2009), 2062–2070.
- [20] D. R. Jocić, Đ. Krtinić, M. Lazarević, *Cauchy-Schwarz inequalities for inner product type transformers in Q^* norm ideals of compact operators*, Positivity **24** (2020), 933–956.
- [21] D. R. Jocić, Đ. Krtinić, M. Sal Moslehian, *Landau and Grüss type inequalities for inner product type integral transformers in norm ideals*, Math. Ineq. Appl. **16** (2013), 109–125.
- [22] D. R. Jocić, M. Lazarević, S. Milošević, *Norm inequalities for a class of elementary operators generated by analytic functions with non-negative Taylor coefficients in ideals of compact operators related to p -modified unitarily invariant norms*, Linear Algebra Appl. **540** (2018), 60–83.
- [23] D. R. Jocić, M. Lazarević, S. Milošević, *Inequalities for generalized derivations of operator monotone functions in norm ideals of compact operators*, Linear Algebra Appl. **586** (2020), 43–63.
- [24] D. R. Jocić, S. Milošević, V. Đurić, *Norm inequalities for elementary operators and other inner product type integral transformers with the spectra contained in the unit disc*, Filomat **31** (2017), 197–206.
- [25] S. Karlin, F. Ost, *Some monotonicity properties of Schur powers of matrices and related inequalities*, Linear Algebra Appl. **68** (1985), 47–65.
- [26] S. Karlin, F. Ost, *Maximal length of common words among random letter sequences*, The Annals of Probability **16** (1988), 535–563.
- [27] S. Karlin, H. Taylor, *A first course in stochastic processes*, Academic, New York, 1975.
- [28] B. Lins, A. Peperko, *Inequalities on the essential joint and essential generalized spectral radius*, Journal of Mathematical Inequalities **18** , (2024), 1489–1514.

- [29] K. Löwner, *Über monotone Matrix Funktionen*, Math. Z. **38** (1934) 177–216.
- [30] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin, 1991.
- [31] D. S. Mitrinović, *Analytic Inequalities*, Springer Verlag, Berlin Heidelberg New York, 1970.
- [32] R. D. Nussbaum, *The radius of essential spectrum*, Duke Math. J. **37** (1970), 473–478.
- [33] A. Peperko, *Spektralni radij integralskih operatorijev in spektralni radij matrik v max-algebri*, Doktorska disertacija, Ljubljana, 2008.
- [34] A. Peperko, *Inequalities for the spectral radius of non-negative functions*, Positivity **13** (2009), 255–272.
- [35] A. Peperko, *Bounds on the generalized and the joint spectral radius of Hadamard products of bounded sets of positive operators on sequence spaces*, Linear Algebra Appl. **437** (2012), 189–201.
- [36] A. Peperko, *Bounds on the joint and generalized spectral radius of the Hadamard geometric mean of bounded sets of positive kernel operators*, Linear Algebra Appl. **533** (2017), 418–427.
- [37] A. Peperko, *Inequalities on the spectral radius, operator norm and numerical radius of the Hadamard weighted geometric mean of positive kernel operators*, Linear Mult. Algebra **67**(8) (2019), 1637–1652.
- [38] A. Peperko, *Inequalities on the joint and generalized spectral and essential spectral radius of the Hadamard geometric mean of bounded sets of positive kernel operators*, Linear Mult. Algebra **67** (2019), 2159–2172.
- [39] A. Peperko, *Inequalities for the spectral radius and essential spectral radius of positive operators on Banach sequence spaces*, Positivity **25** (2021), 1659–1675.
- [40] R. Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag, Berlin, 1960.

- [41] S.-Q. Shen and T.-Z. Huang, *Several inequalities for the largest singular value and the spectral radius of matrices*, Math. Inequal. Appl. **10**(4) (2007), 713–722.
- [42] B. Simon, *Trace ideals and their Applications*, Cambridge University Press, Cambridge, 1979.
- [43] J. G. Stampfli, *Hyponormal operators*, Pacific J. Math. **12** (1962), 1453–1458.
- [44] I. G. Todorov and L. Turowska, *Schur and operator multipliers*, arXiv:0911.0606 [math.FA] , 2009.
- [45] A. C. Zaanen, *Riesz spaces II*, North Holland, Amsterdam, 1983.
- [46] Y. Zhang, *Some spectral norm inequalities on Hadamard products of nonnegative matrices*, Linear Algebra Appl. **556** (2018), 162–170.

Biography

Katarina Bogdanović was born in Belgrade in 1977. She finished elementary school and the First Belgrade High School with the highest grade point average of 5.00. She graduated at the University of Belgrade, Faculty of Mathematics majoring in theoretical and applied mathematics with an average grade of 9.81. During her doctoral studies at the Faculty of Mathematics in Belgrade she was an exchange student in Ljubljana, Slovenia as a part of the Erasmus+ and COST research programs. She was hired as an assistant at the Faculty of Mathematics in Belgrade.

Прилог 1.

Изјава о ауторству

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Број индекса 2013/2017

Студијски програм МАТЕМАТИКА

Наслов рада JOINT SPECTRAL RADIUS OF THE SCHUR-HADAMARD PRODUCT
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Ментор ПРОФ. ДР. ДАНКО ЈОЏИЋ, ДР. АЉОША ПЕРЕРКО

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