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REGULAR GRAPHS WHOSE SECOND LARGEST EIGENVALUE IS AT MOST 1[1]

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Abstract. Regular graphs whose second largest eigenvalue (i.e. λ_2) is at most 1 are considered. Some structural properties of these graphs are obtained, and all these graphs with $\lambda_2 \leq 1$ of degree at most 8 are determined.

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1. Introduction

Let G be a simple graph with n vertices and adjacency matrix A $(= A(G))$. The characteristic polynomial and the eigenvalues of A are also called the *characteristic polynomial* and the *eigenvalues* of G. These eigenvalues, in non-increasing order, are denoted by λ_1 $(= \lambda_1(G)), \ldots, \lambda_n$ $(= \lambda_n(G))$.

The problem of determining the graphs whose second largest eigenvalue is bounded by some (relatively small) number is well studied in the literature. The graphs whose second largest eigenvalue does not exceed $\frac{1}{3}$ or $\sqrt{2}-1$ are determined, while the graphs satisfying $\lambda_2 \leq \frac{\sqrt{5}-1}{2}$ are well characterized but not completely determined (see [8]). In addition, there are many results regarding the cases $\lambda_2 \leq 1$ (see [9] and the references therein) and $\lambda_2 \leq 2$ [8]. More details on this topic can be found in [3, 8], or [5] (including a various bounds on λ_2, its relation with algebraic connectivity or Markov chains, and applications in computer sciences). Here we recall from these references that second largest eigenvalue plays an important role in determining the structure of regular graphs. In particular, it is known that regular graphs with small second largest eigenvalue have more 'round' shape, i.e. smaller diameter and higher connectivity. Moreover, not necessarily regular, but sparse graph having strong connectivity properties is known as an *expander* (for more details, see [7]). Such graphs are relevant to theoretical computer science, the designs of robust computer networks, the theory of error-correcting codes and to complexity theory [7]. Though expanding properties of regular graphs can be measured in several different ways, their common property is a large spectral gap (the

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difference between the degree and the second largest eigenvalue). In this paper we obtain a number of regular graphs with small second largest eigenvalue (and consequently, large spectral gap), and therefore they may be interesting for the application in the above mentioned areas of research.

Regular graphs with all eigenvalues at least -2 are characterized. Namely, they are either complete multipartite graphs with block size 2, line graphs of regular or semiregular bipartite graphs, or embeddable in the Euclidean root system E8 [2]. Further on, regular graphs embeddable in E8 but not of the other types were determined in [1]. Based on these results, another general characterization of regular graphs with all eigenvalues at least -2 is known [4]; we recall this characterization in next section. So, regular graphs with second largest eigenvalue at most 1 can be characterized as their complements, or complements of their disjoint unions. However, such characterization of regular graphs with $\lambda_1 \leq 1$ is rough, and does not provide much information about them, particularly if their complements are disconnected. Hence, our intention is to consider some of those graphs by further analysis, deeper investigation of their structure, and by giving their clear description in some specific cases. We also extend the results obtained in [9] by giving some new general statements, and by considering certain particular cases.

In the next section we fix some notation and mention some results from the literature in order to make the paper more self-contained. The main results are given in Section 4.

2. Preliminaries

A graph consisting of k disjoint copies of an arbitrary graph G will be denoted by kG. A complete graph and a cycle on n vertices will be denoted by K_n and C_n, respectively. A complete bipartite graph with partitions of size m and n is denoted by $K_{m,n}$. The cocktail party graph $CP(n)$ is a unique regular graph with $2n$ vertices of degree $2n-2$ (it is obtained by removal of a perfect matching from K_{2n}). The complement of G we denote by $\overline{G}$, while '$\cup$' stands for the disjoint union of two graphs. For the remaining notation we refer the reader to [3].

The *line graph* $L(H)$ of a graph H is the graph whose vertices are the edges of H, with two vertices in $L(H)$ adjacent whenever the corresponding edges in H have exactly one vertex in common. In this case H is called the *root graph.*

The degree of a regular graph G will be denoted by r_G $(= r)$, while the corresponding graph will be called r-regular. A graph is called (r_1, r_2)-semiregular bipartite, with parameters (n_1, n_2, r_1, r_2), if it is bipartite (i.e. 2-colourable) and vertices in the same colour class have the same degree (n_1 vertices of degree r_1 and n_2 vertices of degree r_2, where $n_1 r_1 = n_2 r_2$). Note that a line graph is regular if and only if its root graph is regular or semiregular bipartite [6].

It is known that each connected regular graph satisfying $\lambda_2 \leq 1$ is a complement of a (not necessarily connected) regular graph whose each component is either

1. a connected regular line graph,

2. a cocktail party graph or

3. one of 187 connected regular exceptional graphs given in [4], pp. 213–227

(see [9, Theorem 3.1]).

Each of 187 connected regular exceptional graphs has between 8 and 28 vertices, while its degree is between 3 and 16. In [4], these graphs are divided into three layers, and if n is the number of vertices, and r the degree of an exceptional graph, then the graphs of the first (resp. the second or the third) layer satisfy

$$(1) \qquad n = 2(r+2) \leq 28 \;\; (\text{resp. } n = \frac{3}{2}(r+2) \leq 27 \text{ or } n = \frac{4}{3}(r+2) \leq 16)$$

Note that any regular exceptional graph belongs to exactly one of layers. This classification will be frequently used in the next section.

3. Main results

In [9], some characterizations of regular graphs with $\lambda_2 \leq 1$ are given, and all r-regular graphs ($r \leq 4$) satisfying $\lambda_2 \leq 1$ are determined. It turns out that some of those results can be generalized to an arbitrary degree, and this is exactly what we do here in Theorem 3.2, and Theorem 3.3. In addition, we give more general characterizations (Lemma 3.1, and Theorem 3.4), and then, using them, we completely determine all r-regular graphs ($5 \leq r \leq 8$) with $\lambda_2 \leq 1$ (Theorem 3.6 – Theorem 3.9).

Clearly, each r-regular ($r \geq 2$) graph with $\lambda_2 \leq 1$ must be connected. Recall that *edge-degree* of an arbitrary edge e is a number of edges adjacent to e. We prove a lemma.

Lemma 3.1. *Let G be a connected r-regular graph on n vertices. If $\overline{G} = L(H)$, where H is a regular graph then $n-r$ is odd and $n-r+1$ divides $4n$.*

Proof. Let H be a regular graph on n edges. Since $r_{\overline{G}} = n-r-1$, we have that each edge of H has degree $n-r-1$, and therefore $r_H = \frac{n-r-1}{2} + 1$. So, $n-r$ must be odd. Next, if N is the number of vertices in H, we get $N = \frac{2n}{r_H}$, and thus $\frac{n-r-1}{2}$ must divide $2n$, that is, $n-r+1$ divides $4n$. This completes the proof. □

Note that we can easily construct some (infinite) families of regular graphs with $\lambda_2 \leq 1$. Namely, it is enough that any of these graphs be a complement of a disjoint union of graphs whose least eigenvalue is not less than -2 (see discussion in Section 1). On the other hand, given an arbitrary regular graph G, then, in general case, the consideration of the structure of G or $\overline{G}$ is not a simple way to answer whether $\lambda_2(G) \leq 1$ or not. In the following two theorems we consider some special cases. Some of resulting graphs appear in [9], as well.

Theorem 3.2. *Let G be an r-regular ($r \geq 4$) graph on $n = 2r$ vertices satisfying $\lambda_2 \leq 1$. We have:*

(i) if $r \neq 6, 7$ then $G = \overline{2K_r}$;

(ii) if $r = 6$ then $G = \overline{2K_6}$ or $G = \overline{L(K_{3,4})}$;

(iii) if $r = 7$ then $G = \overline{2K_7}$ or $G = \overline{L(H)}$, where H is any of two 4-regular graphs with 7 vertices.

Proof. We have $r_{\overline{G}} = r - 1$, so if $\overline{G}$ is disconnected then it has two components with exactly r vertices each. Thus, each of them is a complete graph K_r, giving $G = \overline{2K_r}$. In this way we get (i) and the first parts of (ii), and (iii).

Assume now that $\overline{G}$ is connected. Clearly, it cannot be a cocktail party graph. If $\overline{G}$ is a regular exceptional graph then we get that it either belongs to the second layer, has degree 3, and 6 vertices (which is impossible since a regular exceptional graph in the second layer has at least 9 vertices), or it belongs to the third layer, has degree 2, and 4 vertices (which is also impossible since a regular exceptional graph in the second layer has at least 8 vertices). So, there are no resulting graphs in this case.

It remains that $G = \overline{L(H)}$, where H is regular or semiregular bipartite. If H is regular then, due to Lemma 3.1, r must be odd and $r + 1$ must divide $8r$. In other words, $r + 1$ must divide 8, i.e. $k = 7$. Then we have that H is a regular graph of degree $r_H = \frac{r-1}{2} + 1 = 4$ with $N = \frac{2n}{r_H} = 7$ vertices. There are exactly two such candidates for H), and $G = \overline{L(H)}$. In this way we completed the list of (iii).

Let now H be a semiregular bipartite graph on $n = 2r$ edges. Since each edge of H has degree $r - 1$ we have that H is a (r_1, r_2)-semiregular bipartite, where $r_1 + r_2 = r + 1$. With no loss of generality we can suppose that $r_1 \geq r_2$. If r_2 is equal to 1 we get that $H = 2K_{1,r}$, but then $L(H)$ is not connected. If $r_1 \geq r_2 \geq 2$, then since $r_1, r_2 < r$ and since both r_1 and r_2 must divide $2r$, we easily get that the only solution is $r_1 = \frac{2r}{3}$ and $r_2 = \frac{r}{3} + 1$, giving that $r + 3$ must divide 18. So, $r = 6$, or $r = 15$. The second solution gives a semiregular bipartite graph with parameters $(5, 3, 6, 10)$, which is not possible ($5 > 3$, but we supposed $r_1 \geq r_2$). The first solution gives a rise to the graph $\overline{G} = L(K_{3,4})$. So $G = \overline{L(K_{3,4})}$, which is the remaining graph from (ii), and the proof is complete.

□

We proceed with the following result.

Theorem 3.3. *Let G be an r–regular graph ($r > 17$) on $n > 2r$ vertices satisfying $\lambda_2 \leq 1$. Then $G = \overline{L(K_{2,r+1})}$.*

Proof. If $n > 2r$ then $\overline{G}$ is connected. It can be easily verified that under the assumptions of the theorem $\overline{G}$ cannot be neither a cocktail party graph nor a regular exceptional graph.

Assume that $\overline{G} = L(H)$, where H is a regular graph of degree r_H, and with N vertices. Then we have $N = \frac{2n}{r_H}$, and since $n = 2r_H + r - 1$ it follows that $N = 4 + \frac{2r-2}{r_H}$. Next, we have that $r_H > \frac{r+1}{2}$ (because $n > 2r$), and so $\frac{2r-2}{r_H} < 4 - \frac{8}{r+1} < 4$. It immediately follows that $N \in \{5, 6, 7\}$ (which is not possible since $r_H > \frac{r+1}{2} > 9$).

Let now $\overline{G} = L(H)$, where H is (r_1, r_2)-semiregular bipartite graph on n edges, having n_1 and n_2 vertices in the corresponding colour classes. Obviously, both r_1 and r_2 must be greater than 1. The following facts also hold:

(i) $r_1 n_1 = r_2 n_2 = n$;

(ii) $r_1 + r_2 = n - r + 1$;

(iii) $r_1 r_2 \leq n$.

Without loss of generality we can assume that $r_1 \geq r_2$. So we have $2 \leq r_2 \leq r_1 \leq n - r - 1$, implying that $2(n - r - 1) \leq r_1 r_2 \leq n$. Since $n > 2r$, we get $n = 2r + 1$, or $n = 2r + 2$.

Let $n = 2k + 1$. Then, both r_1 and r_2 must be odd (since they divide n), and r must be even (since $r_1 + r_2 = r + 2$). It then follows that $r_1 r_2 \geq 3(r - 1)$, so for $r > 17$ we have $r_1 r_2 > 2r + 1 = n$, which is not possible.

Let now $n = 2k + 2$. We have $r_1 r_2 \geq 2(r + 1) = n$, and since $r_1 r_2 \leq n$ it immediately follows that $r_1 = 2$, $r_2 = r + 1$. Therefore, $H = K_{2,r+1}$, i.e. $G = \overline{L(K_{2,r+1})}$.

□

It immediately follows from the previous theorem that if the degree r of a regular graph is greater than 17, and the number of its vertices greater than $2r+2$, then such a graph must have $\lambda_2 > 1$. But also if $2 \leq r \leq 17$ and $n > 2r$, besides $L(K_{2,r+1})$, $\overline{G}$ can be the line graph of a regular graph H (if $r \leq 11$) or it can be a regular exceptional graph.

In the first case, we easily get that $n \leq 2r + 3$, with equality holding if $r = 6$, and consequently $H = K_6$ and $G = \overline{L(K_6)}$.

In the second case, using (1), we get that if r-regular graph G with n vertices is the complement of a regular exceptional graph from the first (resp. the second, or the third) layer, then $n = 2(r - 1)$ (resp. $n = 3(r - 1)$, or $n = 4(r - 1)$).

Considering these results and comparing the list of regular exceptional graphs, we get the following theorem.

Theorem 3.4. *Let G be an r-regular graph on n vertices satisfying $\lambda_2 \leq 1$. We have*

(i) if $r > 10$ or $r = 2$, then $n \leq 2r+2$, and if $n = 2r+2$ then $G = \overline{L(K_{2,r+1})}$,

(ii) if $r = 10$ then $n \leq 27$, and if $n = 27$ then G is a complement of the Schläfli graph,

(iii) if $r = 9$ then $n \leq 24$, and if $n = 24$ then G is a complement of a regular exceptional graph on 24 vertices from the second layer,

(iv) *if $r = 8$ then $n \leq 21$, and if $n = 21$ then G is a complement of any of two regular exceptional graphs on 21 vertices from the second layer,*

(v) *if $r = 7$ then $n \leq 18$, and if $n = 18$ then G is a complement of any of four regular exceptional graphs on 18 vertices from the second layer,*

(vi) *if $r = 6$ then $n \leq 15$, and if $n = 15$ then G is a complement of any of six regular exceptional graphs on 15 vertices from the second layer or $G = \overline{L(K_6)}$,*

(vii) *if $r = 5$ then $n \leq 16$, and if $n = 16$ then G is a complement of the Clebsch graph,*

(viii) *if $r = 4$ then $n \leq 12$, and if $n = 12$ then G is a complement of any of five regular exceptional graphs on 12 vertices from the second layer,*

(ix) *if $r = 3$ then $n \leq 10$, and if $n = 10$ then G is the Petersen graph.*

Remark 3.5. Note that the Schläfli, the Clebsch, and the Petersen graph belong to the second, the third, and the first layer, respectively. They can be found in [4].

In Theorem 3.4, we determine the upper bound of n for a fixed r. In addition, we obtain all solutions in the cases when this bound is attained. We now proceed to determine all solutions in some particular cases. It is pointed that all regular graphs satisfying $\lambda_2 \leq 1$ are determined whenever their degree does not exceed 4, so we continue with the next natural step.

Theorem 3.6. *Let G be a 5-regular graph on n vertices satisfying $\lambda_2(G) \leq 1$. Then, G is one of the following graphs: K_6, $\overline{C_8}$, $\overline{C_3 \cup C_5}$, $\overline{C_4 \cup C_4}$, $2K_5$, complement of any of five regular exceptional graphs on 12 vertices from the second layer, complement of the Clebsch graph, $\overline{L(CP(3))}$, or $\overline{L(K_{2,6})}$.*

Proof. Due to Theorem 3.4 (vii), we have $n \leq 16$, and if $n = 16$ then G is a complement of the Clebsch graph.

Let $n < 16$ and assume first that $n \geq 11$. Then, $\overline{G}$ is connected, and since the degree of G is odd, there are two cases to consider. If $\overline{G}$ is an exceptional graph it must have 12 or 14 vertices, and its degree is equal to $n - 6$. The only solutions are five exceptional graphs on 12 vertices from the second layer. Obviously, $\overline{G}$ cannot be a cocktail party graph ($n - r_{\overline{G}} = 6 > 2$).

If $\overline{G}$ is the line graph of a regular graph H, then (due to Lemma 3.1) we have that $n - 4$ divides $4n$, which is not possible if $n = 14$. If $n = 12$, then the degree of H is $r_H = \frac{12-6}{2} + 1 = 4$, and the number of vertices of H is $N = \frac{2n}{r_H} = 6$, so $H = CP(3)$, $\overline{G} = L(CP(3))$, and finally, $G = \overline{L(CP(3))}$.

If $\overline{G}$ is the line graph of a semiregular bipartite graph, since $n > 2r_G$ we get (similarly to Theorem 3.3) that n must be equal to 12, and $G = \overline{L(K_{2,6})}$.

Assume now that $n < 11$. If $n = 10$, then, due to Theorem 3.2, we have that $G = 2K_5$. If $n = 8$ then $r_{\overline{G}} = 2$, which yields that $\overline{G}$ is either a cycle or it is a disconnected graph whose all components are cycles. Additionally, if

$\overline{G} = C_{k_1} \cup C_{k_2} \cup \ldots \cup C_{k_l}$, then $k_1 + k_2 + \ldots + k_l = n$ holds. So, in this case we have $G = \overline{C_8}$ or $G = \overline{C_3 \cup C_5}$ or $G = \overline{C_4 \cup C_4}$. Finally, if $n = 6$ then $G = K_6$.

Collecting the graphs obtained, we get the above list.

□

Theorem 3.7. *Let G be a 6-regular graph on n vertices satisfying $\lambda_2(G) \leq 1$. Then, G is one of the following graphs: K_7, $CP(4)$, $\overline{C_9}$, $\overline{C_3 \cup C_6}$, $\overline{C_4 \cup C_5}$, $\overline{C_3 \cup C_3 \cup C_3}$, $\overline{L(K_4) \cup L(K_{2,3})}$, complement of any of five regular exceptional graphs on 10 vertices from the first layer, $\overline{CP(3) \cup K_5}$, $\overline{L(K_{3,4})}$, $\overline{L(2K_{1,6})}$, complement of any of six regular exceptional graphs on 15 vertices from the second layer, $\overline{L(K_6)}$, or $\overline{L(K_{2,7})}$.*

Proof. Due to Theorem 3.4 (vi), we have $n \leq 15$, and if $n = 15$ then G is a complement of any of six regular exceptional graphs on 18 vertices from the second layer or $G = \overline{L(K_6)}$.

Let $n < 15$ and assume first that $n \geq 13$. Then, $\overline{G}$ is connected, and we have two cases: $n = 13$ and $n = 14$. If $\overline{G}$ is an exceptional graph its degree will be $r_{\overline{G}} = 6$ if $n = 13$, or $r_{\overline{G}} = 7$ if $n = 14$, but there are no such regular exceptional graphs. It is obvious that $\overline{G}$ cannot be a cocktail party graph since $n - r_{\overline{G}} = 7 > 2$. Since $n - 4$ does not divide $4n$ for $n \in \{13, 14\}$, due to Lemma 3.1, $\overline{G}$ cannot be the line graph of a regular graph. If $\overline{G}$ is the line graph of a semiregular bipartite graph, since $n > 2r_G$, n must be equal to 14, and $G = \overline{L(K_{2,7})}$ (see the proof of Theorem 3.3).

Assume now that $n < 13$. If $n = 12$, then, due to Theorem 3.2, we have that $G = 2K_6$, or $G = \overline{L(K_{3,4})}$.

If $n = 11$, we have that $r_{\overline{G}} = 4$. If a component of $\overline{G}$ is an exceptional graph then the number of vertices in that component would be at least 8 (according to its degree). So $\overline{G}$ would have to be a connected graph, but there are no regular exceptional graphs of degree 4 on 11 vertices.

If a component of $\overline{G}$ is the line graph of a regular graph H, then H has $m \leq 11$ edges, and its degree is $r_H = \frac{r_{\overline{G}}}{2} + 1 = 3$. If N is the number of vertices in H, then we have $3N = 2m$, so 3 divides $2m$. The possibilities are: $m = 9$ (but the other components of $\overline{G}$ would then have 2 or less vertices), $m = 6$ (then $H = K_4$, so $L(H) = CP(3)$, while the other component of $\overline{G}$ has 5 vertices, which yields that $G = \overline{CP(3) \cup K_5}$), or $m = 3$ (but this is not possible since then we have $N = 2$ and $r_H = 3$).

If a component of $\overline{G}$ is the line graph of an (r_1, r_2)-semiregular bipartite graph then the sum $r_1 + r_2$ is equal to $r_{\overline{G}} + 2 = 6$. By inspecting all possibilities we get that $K_{1,5}$ is the only solution. Then, the other component of $\overline{G}$ has 6 vertices, and since its degree is 4, it must be $CP(3)$. The solution we got is the same as in the previous case, and the same one arises if we suppose that one component of $\overline{G}$ is a cocktail party graph.

If $n = 10$, then $r_{\overline{G}} = 3$. If a component of $\overline{G}$ is a regular exceptional graph then, according to its degree, the number of vertices in that component must be 10, so $\overline{G}$ is connected, and it is one of five cubic exceptional graphs on 10 vertices from the first layer.

The cocktail party graph cannot be a component of $\overline{G}$, because its degree is odd. The same argument gives us that it also cannot be the line graph of a regular graph H since the degree of H is $r_H = \frac{r_{\overline{G}}}{2} + 1$.

Thus, if $\overline{G}$ is disconnected, all of its components must be line graphs of semiregular bipartite graphs, and if a component of $\overline{G}$ is a line graph of (r_1, r_2)-semiregular bipartite graph then the sum $r_1 + r_2$ is equal to $r_{\overline{G}} + 2 = 5$. The only solution are two components: one having 4 vertices ($r_1 = 1$, $r_2 = 4$), and the other having 6 ($r_1 = 2$, $r_2 = 3$), i.e. $\overline{G} = L(K_{1,4}) \cup L(K_{2,3})$ or $G = \overline{L(K_4) \cup L(K_{2,3})}$.

If $\overline{G}$ is a connected line graph of an (r_1, r_2)-semiregular bipartite graph then we will have $r_1 + r_2 = 5$, and both r and s divide 10, which is not possible.

If $n = 9$, then, similarly to the previous theorem, we have that $\overline{G}$ must be a cycle, or a disconnected graph whose all components are cycles, and so G must be $\overline{C_9}$, $\overline{C_3 \cup C_6}$, $\overline{C_4 \cup C_5}$ or $\overline{C_3 \cup C_3 \cup C_3}$.

Finally, if $n = 8$, then $G = CP(4)$, while if $n = 7$ then $G = K_7$.

Collecting the graphs obtained, we get the above list.

□

The proofs of the following two theorems are very similar to the proofs of Theorem 3.6 and Theorem 3.7, and therefore they will be omitted.

Theorem 3.8. *Let G be a 7-regular graph on n vertices satisfying $\lambda_2(G) \leq 1$. Then, G is one of the following graphs: K_8, $\overline{C_{10}}$, $\overline{C_3 \cup C_7}$, $\overline{C_4 \cup C_6}$, $\overline{C_5 \cup C_5}$, $\overline{C_3 \cup C_3 \cup C_4}$, complement of any of eight regular exceptional graphs on 12 vertices from the first layer, $\overline{2CP(3)}$, $\overline{L(H_i)}$ (where H_i $(i = 1, \ldots, 5)$, are all regular cubic graphs on 8 vertices), $\overline{L(B_1)}$ (where B_1 is semiregular bipartite graph with parameters $(3, 6, 4, 2)$), $\overline{L(B_2)}$ (where B_2 is semiregular bipartite graph with parameters $(4, 4, 3, 3)$), $G = \overline{2K_7}$, $G = \overline{L(H)}$ (where H is any of two regular graphs of degree 4 with 7 vertices), complement of any of 38 regular exceptional graphs on 18 vertices from the second layer, or $\overline{L(K_{2,8})}$.*

Theorem 3.9. *Let G be an 8-regular graph on n vertices satisfying $\lambda_2(G) \leq 1$. Then, G is one of the following graphs: K_9, $CP(5)$, $\overline{C_{11}}$, $\overline{C_3 \cup C_8}$, $\overline{C_4 \cup C_7}$, $\overline{C_5 \cup C_6}$, $\overline{C_3 \cup C_3 \cup C_5}$, $\overline{C_3 \cup C_4 \cup C_4}$, $\overline{3K_4}$, $\overline{L(2K_{2,3})}$, $\overline{H \cup K_5}$ (where H is a regular exceptional graph on 8 vertices from the third layer), $\overline{L(K_{2,4}) \cup K_5}$, complement of any of 21 regular exceptional graphs on 14 vertices from the first layer, $\overline{CP(4) \cup K_7}$, $\overline{L(K_{3,5})}$, $\overline{2K_8}$, complement of any of two regular exceptional graphs on 21 vertices from the second layer, or $\overline{L(K_{2,9})}$.*

References

[1] Bussemaker, F.C., Cvetković, D.M., Seidel, J.J., Graphs related to exceptional root systems. In: Combinatorics I, Proceedings of the 5th Hungarian Colloquium on Combinatorics, Keszthely, 1976. (Hajnal A., Sos V.T., eds.), pp. 185–191. Amsterdam: North-Holland 1978.

[2] Cameron, P.J., Goethals, J.M., Seidel, J.J., Shult, E.E., Line graphs, root systems, and elliptic geometry. J. Algebra, 43 (1976), 305–327.

[3] Cvetković, D., Doob, M., Sachs, H., Spectra of graphs – theory and application. Heidelberg and Leipzig: Johann Ambrosius Barth Verlag 1995.

[4] Cvetković, D., Rowlinson, P., Simić, S., Spectral generalizations of line graphs – On line graphs with least eigenvalue -2. Cambridge: University Press 2004.

[5] Cvetković, D., Simić, S., The second largest eigenvalue of a graph (a survey). Filomat, 9 (1995), 449–472.

[6] Harary, F., Graph theory. Reading: Addison–Wesley Publ. Comp. 1969.

[7] Hoory, S., Linial, N., Wigderson, A., Expander graphs and their applications. Bull. Amer. Math. Soc., 43 (2006), 439–561.

[8] Petrović, M., Radosavljević, Z., Spectrally constrained graphs. Kragujevac: Faculty of Science 2001.

[9] Stanić, Z., On regular graphs and coronas whose second largest eigenvalue does not exceed 1. Linear Multilinear Algebra, 58 (2010), 545–554.

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