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SOME PROPERTIES OF THE EIGENVALUES OF THE NET LAPLACIAN MATRIX OF A SIGNED GRAPH

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Abstract

Given a signed graph $\dot{G}$, let $A_{\dot{G}}$ and $D^{\pm}_{\dot{G}}$ denote its standard adjacency matrix and the diagonal matrix of vertex net-degrees, respectively. The net Laplacian matrix of $\dot{G}$ is defined to be $N_{\dot{G}} = D^{\pm}_{\dot{G}} - A_{\dot{G}}$. In this study we give some properties of the eigenvalues of $N_{\dot{G}}$. In particular, we consider their behaviour under some edge perturbations, establish some relations between them and the eigenvalues of the standard Laplacian matrix and give some lower and upper bounds for the largest eigenvalue of $N_{\dot{G}}$.

Keywords: (net) Laplacian matrix, edge perturbations, largest eigenvalue, net-degree.

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1. Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is an 'unsigned' graph, called the *underlying graph*, and $\sigma \colon E \longrightarrow \{-1, +1\}$ is the *sign function*. The edge set of a signed graph is composed of the subset of positive edges E^+ and the subset of negative edges E^-. Throughout the paper we interpret a graph as a signed graph with all the edges being positive. We denote the number of vertices of a signed graph by n.

The *degree* d_i of a vertex i of $\dot{G}$ is the number of its neighbours. The *positive degree* d_i^+ is the number of positive neighbours of i (i.e., those adjacent to i by a positive edge). In the similar way, we define the *negative degree* d_i^-. The *net-degree* of i is defined to be $d_i^{\pm} = d_i^+ - d_i^-$.

The adjacency matrix $A_{\dot{G}}$ is obtained from the standard adjacency matrix of its underlying graph by reversing the sign of all 1s that correspond to negative edges. The Laplacian matrix is defined to be $L_{\dot{G}} = D_{\dot{G}} - A_{\dot{G}}$, where $D_{\dot{G}}$ is the diagonal matrix of vertex degrees; clearly, $D_{\dot{G}}$ coincides with the diagonal matrix of vertex degrees of the underlying graph G. The *net Laplacian matrix* is $N_{\dot{G}} = D^{\pm}_{\dot{G}} - A_{\dot{G}}$, where $D^{\pm}_{\dot{G}}$ is the diagonal matrix of vertex net-degrees. We denote the eigenvalues (with repetition) of $A_{\dot{G}}$, $L_{\dot{G}}$ and $N_{\dot{G}}$ by $\lambda_1, \lambda_2, \dots, \lambda_n$, $\mu_1, \mu_2, \dots, \mu_n$ and $\nu_1, \nu_2, \dots, \nu_n$, respectively. In the majority of this paper we also assume that they are indexed non-increasingly. An exception occurs in the forthcoming Lemma 6. To ease language, in the sequel we abbreviate the spectrum, the eigenvalues and the eigenvectors of $N_{\dot{G}}$ as the *spectrum*, the *eigenvalues* and the *eigenvectors* of $\dot{G}$.

A significance of the spectrum of the net Laplacian matrix in control theory was recognized in [3]. The same topic is considered from a graph theoretic perspective in [6]. (For a nice introduction to the controllability of dynamical systems based on graph matrices, we refer the reader to [4].) In [7] we considered some advantages of using the net Laplacian matrix instead of the Laplacian matrix (in study of signed graphs). In this paper we continue our research on the eigenvalues of $N_{\dot{G}}$. Apart from some particular results, we consider how they change when we apply some standard edge perturbations and give certain relations between them and the eigenvalues of $L_{\dot{G}}$. We pay a special attention to the largest eigenvalue and derive a formula for it (based on the Rayleigh principle), which in fact gives a way for constructing lower bounds for this eigenvalue. We also establish an upper bound expressed in terms of certain structural parameters.

In Section 2 we give some terminology and notation. Our contribution is reported in Sections 3 and 4.

2. Preliminaries

We use $\mathbf{j}$ and $\mathbf{0}$ to denote the all-1 and the all-0 vector, respectively, and I and J to denote the identity and the all-1 matrix, respectively. We say that a signed graph is *bipartite* or *regular* if the same holds for its underlying graph. A signed graph is said to be *net-regular* if the net-degree is a constant on the vertex set. A *trivial* (signed) graph consists of a single vertex. The *negation* $-\dot{G}$ of $\dot{G}$ is obtained by reversing the sign of all edges of $\dot{G}$.

By N_i we denote the (open) neighbourhood of a vertex i. A *walk* in a signed graph is defined in the same way as the walk in a graph. A walk is *positive* if the number of negative edges contained (counted with their repetition) is even; otherwise, it is *negative*. In particular, we use $w_2(i, j)$ to denote the difference between the numbers of positive and negative walks of length 2 starting at i and terminating at j. The number of positive (respectively, negative) walks of length 2 between the same vertices is denoted by $w_2^+(i, j)$ (respectively, $w_2^-(i, j)$).

The *vertex connectivity* $c_v(\dot{G})$ of $\dot{G}$ is equal to the vertex connectivity of G, and so it is the minimum number of vertices whose removal results in a trivial or disconnected signed graph.

If $\mathbf{x} = (x_1, x_2, \dots, x_n)^\intercal$ is an eigenvector associated with an eigenvalue ν (of $N_{\dot{G}}$), then the *eigenvalue equation* related to ν at vertex i reads

$$(d_i^\pm - \nu)x_i = \sum_{ij \in E(\dot{G})} \sigma(ij)x_j, \tag{1}$$

where, as defined in the first section, $\sigma(ij)$ is the value of the sign function σ on ij. Conversely, if (1) holds for some non-zero vector $\mathbf{x}$, real number ν and all the vertices of $\dot{G}$, then ν is an eigenvalue of $\dot{G}$ and $\mathbf{x}$ is an associated eigenvector.

3. General Results

Observe that 0 is an eigenvalue of $N_{\dot{G}}$ for every signed graph $\dot{G}$. The corresponding eigenspace contains the all-1 vector $\mathbf{j}$. In general, $N_{\dot{G}}$ can have both positive and negative eigenvalues. Here is a necessary condition for the non-existence of negative ones.

Lemma 1. *If the eigenvalues of a connected signed graph $\dot{G}$ are non-negative, then $d_i^\pm > 0$, for $1 \leq i \leq n$.*

Proof. We assume to the contrary and use the Sylvester's criterion which states that a Hermitian matrix is positive semidefinite if and only if all principal minors are non-negative.

For $d_j^\pm < 0$, the corresponding minor (of the 1×1 principal submatrix) is negative, and so $\dot{G}$ has a negative eigenvalue. For $d_j^\pm = 0$, since $\dot{G}$ is connected, there is a vertex, say u, adjacent to j. With a suitable labelling of vertices, we get that

$$\det \begin{pmatrix} 0 & -\sigma(ju) \\ -\sigma(ju) & d_u^\pm \end{pmatrix}$$

is a minor of the corresponding matrix. Since it is negative, we complete the proof. ∎

We proceed with a Fiedler-like formula (cf. [2]) based on the coordinates of an associated eigenvector.

Theorem 2. *For the largest eigenvalue ν of $N_{\dot{G}}$ associated with a non-constant eigenvector, we have*

$$\nu(\dot{G}) = 2n \max_{\mathbf{x}\neq\mathbf{0},\langle\mathbf{x},\mathbf{j}\rangle=0} \frac{\sum_{ij\in E^+(\dot{G})}(x_i-x_j)^2 - \sum_{ij\in E^-(\dot{G})}(x_i-x_j)^2}{\sum_{i,j\in V(\dot{G})}(x_i-x_j)^2}. \tag{2}$$

Proof. According to the Rayleigh principle, we have

$$\nu(\dot{G}) = \max_{\mathbf{x}\neq\mathbf{0},\langle\mathbf{x},\mathbf{j}\rangle=0} \frac{\mathbf{x}^T N_{\dot{G}}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$

Computing the numerator, we get

$$\mathbf{x}^T N_{\dot{G}}\mathbf{x} = \sum_{i=1}^{n} d_i^{\pm} x_i^2 - 2\sum_{ij\in E(\dot{G})}\sigma(ij)x_ix_j.$$

Since

$$\sum_{i=1}^{n} d_i^{\pm} x_i^2 = \sum_{ij\in E^+(\dot{G})}(x_i^2+x_j^2) - \sum_{ij\in E^-(\dot{G})}(x_i^2+x_j^2)$$

and

$$2\sum_{ij\in E(\dot{G})}\sigma(ij)x_ix_j = 2\left(\sum_{ij\in E^+(\dot{G})}x_ix_j - \sum_{ij\in E^-(\dot{G})}x_ix_j\right),$$

we deduce

$$\mathbf{x}^T N_{\dot{G}}\mathbf{x} = \sum_{ij\in E^+(\dot{G})}(x_i-x_j)^2 - \sum_{ij\in E^-(\dot{G})}(x_i-x_j)^2. \tag{3}$$

For the denominator, we have

$$\mathbf{x}^T\mathbf{x} = \sum_{i=1}^{n} x_i^2 = \frac{1}{n}\left(\frac{1}{2}\sum_{i,j\in V(\dot{G})}(x_i-x_j)^2 + \left(\sum_{i=1}^{n} x_i\right)^2\right),$$

where the latter equality follows by the Lagrange's identity. Since $\langle\mathbf{x},\mathbf{j}\rangle = 0$, we have $\sum_{i=1}^{n} x_i = 0$, and thus

$$\mathbf{x}^T\mathbf{x} = \frac{1}{2n}\sum_{i,j\in V(\dot{G})}(x_i-x_j)^2. \tag{4}$$

Now, (2) is completed by (3) and (4). ■

Since $N_{-\dot{G}} = -N_{\dot{G}}$, the least eigenvalue of $N_{\dot{G}}$ associated with a non-constant eigenvector is obtained by replacing max with min in (2).

Example 3.1. Clearly, if $\nu(\dot{G}) > 0$, then the assumption of Theorem 2 that an associated eigenvector is non-constant is satisfied automatically. Observe that the possibility $\nu(\dot{G}) = 0$ may occur, and the least eigenvalue associated with a non-constant eigenvector can also be zero. Indeed, the eigenvalues of the signed graph $\dot{G}$ determined by

$$N_{\dot{G}} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

are 0 with multiplicity 2 and -4 with multiplicity 2. Thus, its largest eigenvalue is 0 and has a non-constant eigenvector. For the least eigenvalue equal to 0, we consider the negation of $\dot{G}$.

This example is not obtained accidentally, since by [7], if $\dot{G}_1 \nabla^- \dot{G}_2$ is the negative join of $\dot{G}_1$ and $\dot{G}_2$ (i.e., $\dot{G}_1 \nabla^- \dot{G}_2$ is obtained by inserting a negative edge between every vertex of $\dot{G}_1$ and every vertex of $\dot{G}_2$) and the number of vertices of $\dot{G}_2$ appear as an eigenvalue of $\dot{G}_1$, then 0 is an eigenvalue of $\dot{G}_1 \nabla^- \dot{G}_2$ with multiplicity at least 2.

Since $N_{\dot{G}}$ is similar to a diagonal matrix with the eigenvalues on the diagonal, we deduce that all the eigenvalues of $\dot{G}$ are zero if and only if $E(\dot{G}) = \emptyset$. For $E(\dot{G}) \neq \emptyset$, up to taking the negation of a signed graph, we can always assure that $\nu(\dot{G})$ of Theorem 2 is the largest eigenvalue $\nu_1(\dot{G})$. In fact, this theorem gives a tool for computing lower bounds for $\nu_1(\dot{G})$ as any choice for $\mathbf{x}$ gives one of them.

In what follows we consider the behaviour of the eigenvalues under certain edge perturbations.

Lemma 3. *For a signed graph $\dot{G}$, if $\dot{H}$ is obtained from $\dot{G}$ either by*

(i) *adding at least one positive edge,*

(ii) *removing at least one negative edge, or*

(iii) *reversing the sign of at least one negative edge,*

then we have

$$\nu_i(\dot{H}) \geq \nu_i(\dot{G}), \tag{5}$$

for $1 \leq i \leq n$. The inequality is strict for at least one i.

Proof. For (i) and (ii) we have $N_{\dot{H}} = N_{\dot{G}} + L_F$, where L_F is the Laplacian matrix of a graph F induced either by positive edges added to $\dot{H}$ or negative edges removed from $\dot{G}$. Using the Courant-Weyl inequalities [5, Theorem 1.3], we get $\nu_i(\dot{H}) \geq \nu_i(\dot{G}) + \mu_n(F) = \nu_i(\dot{G}) + 0$, which gives (5).

For (iii), let L_F be the Laplacian matrix of a graph induced by negative edges whose sign is reversed. Then $N_{\dot{H}} = N_{\dot{G}} + 2L_F$, and (5) follows in the same way as before.

Since in all three cases the trace of $\dot{H}$ is strictly greater than the trace of $\dot{G}$, we conclude that the inequality is strict for at least one i. ∎

Here is a natural consequence.

Corollary 3.1. *For a signed graph $\dot{G}$, we have*

$$\nu_i(\dot{G}) \leq \mu_i(G),$$

for $1 \leq i \leq n$, where G is its underlying graph. If $\dot{G} \ncong G$ (that is, if $\dot{G}$ is non-isomorphic to its underlying graph), then the inequality is strict for at least one i.

Proof. The result follows since G is obtained from $\dot{G}$ by the operation described in Lemma 3(iii). ∎

Now, we consider a relation between the net Laplacian eigenvalues and the Laplacian eigenvalues.

Lemma 4. *For a signed graph $\dot{G}$, we have*

$$\nu_i(\dot{G}) \leq \mu_i(\dot{G}), \tag{6}$$

for $1 \leq i \leq n$. If $\dot{G} \ncong G$, then the inequality is strict for at least one i. If every vertex of $\dot{G}$ is incident with at least one negative edge, then the inequality is strict for every i.

Proof. Let $D^-_{\dot{G}}$ be the diagonal matrix of negative vertex degrees. It holds $L_{\dot{G}} = N_{\dot{G}} + 2D^-_{\dot{G}}$, and so

$$\mu_i(\dot{G}) \geq \nu_i(\dot{G}) + \delta_n\left(D^-_{\dot{G}}\right), \tag{7}$$

where $\delta_n\big(D^-_{\dot{G}}\big)$ denotes the least eigenvalue of $D^-_{\dot{G}}$. Since $D^-_{\dot{G}}$ is diagonal dominant with non-negative main diagonal, it is positive semidefinite, which yields $\lambda_n\big(D^-_{\dot{G}}\big) \geq 0$, and we get (6).

If $\dot{G} \ncong G$, considering traces of $N_{\dot{G}}$ and $L_{\dot{G}}$, we get the strict inequality for at least one i.

If every vertex of $\dot{G}$ is incident with at least one negative edge, the main diagonal of $D^-_{\dot{G}}$ is positive, which means that $\delta_n\big(D^-_{\dot{G}}\big) > 0$, and this together with (7) gives the assertion. ∎

Obviously, if $\dot{G}$ is net-regular with net degree $d^{\pm}$, then $N_{\dot{G}} = d^{\pm}I - A_{\dot{G}}$, which means that $\nu_i(\dot{G}) = d^{\pm}I - \lambda_i(\dot{G})$, for $1 \leq i \leq n$. In what follows we consider signed graphs with constant negative vertex degree; i.e., those with $d_i^- = const.$ The following definitions are needed.

For a signed graph $\dot{G}$, we introduce the vertex-edge *orientation* $\eta\colon V(\dot{G}) \times E(\dot{G}) \longrightarrow \{1, 0, -1\}$ formed by obeying the following rules: (1) $\eta(i, jk) = 0$ if $i \notin \{j, k\}$, (2) $\eta(i, ij) = 1$ or $\eta(i, ij) = -1$ and (3) $\eta(i, ij)\eta(j, ij) = -\sigma(ij)$. Then $\dot{G}_\eta$ consists of $\dot{G}$ together with the orientation, so it is the pair $(\dot{G}, \eta)$. The (vertex-edge) *incidence matrix* B_η is the matrix whose rows and columns are indexed by $V(\dot{G})$ and $E(\dot{G})$ respectively, such that its (i, e)-entry is equal to $\eta(i, e)$.

Note that, regardless of the orientation chosen, we have $B_\eta B_\eta^\intercal = L_{\dot{G}}$. Similarly, we have $B_\eta^\intercal B_\eta = 2I + A_{L(\dot{G}_\eta)}$, where $L(\dot{G}_\eta)$ is taken to be the *signed line graph* of $\dot{G}_\eta$. It is not difficult to show that signed line graphs obtained by different orientations share the same spectrum (since they are switching equivalent [8]), so we may say that, up to the switching equivalence, there is a unique signed line graph $L(\dot{G})$ of $\dot{G}$.

Lemma 5. *If a signed graph $\dot{G}$ with n vertices has a constant negative vertex degree d^-, then*

(i) $\nu_i(\dot{G}) = \mu_i(\dot{G}) - 2d^-$, *for* $1 \leq i \leq n$;

(ii) *apart from possible eigenvalue $-2d^-$ of $N_{\dot{G}}$ and -2 of $A_{L(\dot{G})}$, the eigenvalues (with repetitions) of $N_{\dot{G}}$ and $A_{L(\dot{G})}$ coincide.*

Proof. For (i) we have $N_{\dot{G}} = L_{\dot{G}} - 2d^-I$, which leads to the conclusion.

For (ii), since $B_\eta B_\eta^\intercal$ and $B_\eta^\intercal B_\eta$ share the same non-zero eigenvalues, we get the assertion. ■

Of course, if $\dot{G}$ is regular and net-regular, then the negative vertex degree is constant on the vertex set, and we have $N_{\dot{G}} = d^{\pm}I - A_{\dot{G}} = L_{\dot{G}} - 2d^-I$. In other words, the eigenvalues of $N_{\dot{G}}$ are fully determined by the eigenvalues of $A_{\dot{G}}$ and also by those of $L_{\dot{G}}$.

We conclude this section by considering how the eigenvalues are affected upon the removal of specified vertices.

Lemma 6. *Assume that a signed graph $\dot{H}$ with $n+k$ vertices contains a set S of k vertices, such that none of them is incident with a negative edge. If $\dot{G}$ is obtained by removing the vertices in S and $\nu_1(\dot{G}) \geq \nu_2(\dot{G}) \geq \cdots \geq \nu_{n-1}(\dot{G}), \nu_n(\dot{G}) = 0$ are its eigenvalues, then there exist $n-1$ eigenvalues of $\dot{H}$, $\nu_{j_1}(\dot{H}) \geq \nu_{j_2}(\dot{H}) \geq \cdots \geq \nu_{j_{n-1}}(\dot{H})$, such that $\nu_{j_i}(\dot{H}) \leq \nu_i(\dot{G}) + k$, for $1 \leq i \leq n-1$.*

In particular, $\nu_{j_{n-1}}(\dot{H})$ may be taken to be the least eigenvalue associated with a non-constant eigenvector.

Proof. Let $\dot{H}'$ be obtained from $\dot{H}$ by adding all possible positive edges, such that at least one endpoint of each of them belongs to S. Starting by labelling the vertices in S and then those outside S, we get

$$N_{\dot{H}'} = \begin{pmatrix} N_{\dot{G}} + kI_{n\times n} & -J_{n\times k} \\ -J_{k\times n} & (n+k)I_{k\times k} - J_{k\times k} \end{pmatrix}.$$

Therefore, if $\mathbf{x_i}$ is a non-constant eigenvector associated with $\nu_i(\dot{G})$, then

$$N_{\dot{H}'} \begin{pmatrix} \mathbf{x_i} \\ \mathbf{0} \end{pmatrix} = (\nu_i(\dot{G}) + k) \begin{pmatrix} \mathbf{x_i} \\ \mathbf{0} \end{pmatrix},$$

as $\langle \mathbf{x_i}, \mathbf{j} \rangle = 0$. Thus, $\nu_i(\dot{G}) + k$ is an eigenvalue of $\dot{H}'$. Hence, we may denote $\nu_{j_i}(\dot{H}') = \nu_i(\dot{G}) + k$, for $1 \leq i \leq n-1$. By Lemma 3(i), we have $v_{j_i}(\dot{H}) \leq v_{j_i}(\dot{H}')$, for $1 \leq i \leq n-1$, which gives the assertion.

The particular case follows since the least eigenvalue of $\dot{H}$ associated with a non-constant eigenvector does not exceeds $\nu_{n-1}(\dot{H})$. ∎

We extend a well-known result attributed to Fiedler [1].

Corollary 3.2. *For a non-complete signed graph $\dot{G}$ with n vertices, we have $\nu_{n-1}(\dot{G}) \leq c_v(\dot{G})$, where $c_v(\dot{G})$ denotes the vertex connectivity of $\dot{G}$.*

Proof. Clearly, we may assume that $\nu_{n-1}(\dot{G}) \geq 0$; otherwise, the statement is trivial.

First, if $\dot{G}$ is disconnected, then $c_v(\dot{G}) = 0$ and also $\nu_{n-1}(\dot{G}) = 0$ (as 0 is an eigenvalue of multiplicity at least 2), and we are done.

Assume further that $\dot{G}$ is connected, and let U denote a subset of vertices whose removal disconnects $\dot{G}$ and such that $c_v(\dot{G}) = |U|$. By Lemma 3(iii), $\nu_{n-1}(\dot{G}) \leq \nu_{n-1}(\dot{H})$, where $\dot{H}$ is obtained by reversing the sign of every negative edge with at least one endpoint in U. Obviously, $c_v(\dot{H}) = |U|$. If $\dot{H}'$ is obtained from $\dot{H}$ by removing all vertices of U, then $\dot{H}'$ is disconnected (since $\dot{G}$ is non-complete), and so its least eigenvalue associated with a non-constant eigenvector is non-positive. By Lemma 6, we have $\nu_{n-1}(\dot{H}) \leq c_v(\dot{H})$, which gives the assertion. ∎

4. An Upper Bound for ν_1

Here we derive an upper bound for the largest eigenvalue ν_1.

Theorem 7. *For a connected signed graph $\dot{G}$,*

$$\nu_1 \leq \max\left\{\frac{1}{2}\left(d_i^{\pm} + \sqrt{{d_i^{\pm}}^2 + 4(m_i^{\pm} + n_i)}\right) \;:\; 1 \leq i \leq n\right\}, \tag{8}$$

where, for a vertex i, $d_i^{\pm}$ denotes its net-degree, $m_i^{\pm} = \sum_{j\sim i} |d_j^{\pm}|$ and $n_i = \sum_{j\sim i}(d_j + |w_2^+(i,j) - w_2^-(i,j)| - |N_i \cap N_j|)$.

Equality holds if $\dot{G}$ is regular with all edges being positive.

Proof. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^\intercal$ be an eigenvector associated with the largest eigenvalue ν_1 of the net Laplacian matrix $N_{\dot{G}}$ of $\dot{G}$. Let x_i be the largest coordinate of $\mathbf{x}$ in absolute terms; without loss of generality, we may assume that x_i is positive, and then we have $|x_j| \le x_i$, for $1 \le j \le n$.

Since $\nu_1\mathbf{x} = N_{\dot{G}}\mathbf{x} = (D_{\dot{G}}^{\pm} - A_{\dot{G}})\mathbf{x}$, we have

$$\begin{aligned}\nu_1^2\mathbf{x} &= (D_{\dot{G}}^{\pm} - A_{\dot{G}})^2\mathbf{x}\\ &= D_{\dot{G}}^{\pm 2}\mathbf{x} - D_{\dot{G}}^{\pm}A_{\dot{G}}\mathbf{x} - A_{\dot{G}}D_{\dot{G}}^{\pm}\mathbf{x} + A_{\dot{G}}^2\mathbf{x}.\end{aligned}$$

In particular,

$$\nu_1^2 x_i = d_i^{\pm 2}x_i - d_i^{\pm}\sum_{ij\in E(\dot{G})}\sigma(ij)x_j - \sum_{ij\in E(\dot{G})} d_j^{\pm}\sigma(ij)x_j + \sum_{\substack{k=i \text{ or}\\ d(k,i)=2}} w_2(i,k)x_k.$$

By (1), we get

$$\nu_1^2 x_i = d_i^{\pm 2}x_i - d_i^{\pm}(d_i^{\pm} - \nu_1)x_i - \sum_{ij\in E(\dot{G})} d_j^{\pm}\sigma(ij)x_j + \sum_{\substack{k=i \text{ or}\\ d(k,i)=2}} w_2(i,k)x_k,$$

i.e.,

$$\big(\nu_1^2 - d_i^{\pm}\nu_1\big)x_i = -\sum_{ij\in E(\dot{G})} d_j^{\pm}\sigma(ij)x_j + \sum_{\substack{k=i \text{ or}\\ d(k,i)=2}} w_2(i,k)x_k. \tag{9}$$

Considering the first term of the right-hand side, we get

$$-\sum_{ij\in E(\dot{G})} d_j^{\pm}\sigma(ij)x_j \le \left|\sum_{ij\in E(\dot{G})} d_j^{\pm}\sigma(ij)x_j\right| \le \sum_{ij\in E(\dot{G})} |d_j^{\pm}|x_i = m_i^{\pm}x_i. \tag{10}$$

For the second term, we have

$$\begin{aligned}\sum_{\substack{k=i \text{ or}\\ d(k,i)=2}} w_2(i,k)x_k &= \sum_{\substack{k=i \text{ or}\\ d(k,i)=2}} w_2^+(i,k)x_k - \sum_{\substack{k=i \text{ or}\\ d(k,i)=2}} w_2^-(i,k)x_k\\ &= \sum_{ik\in E(\dot{G})} w_2^+(i,k)x_k + \sum_{ik\notin E(\dot{G})} w_2^+(i,k)x_k\\ &\quad - \sum_{ik\in E(\dot{G})} w_2^-(i,k)x_k - \sum_{ik\notin E(\dot{G})} w_2^-(i,k)x_k.\end{aligned}$$

We consider the terms on the right-hand side of the previous equality. For the first and the third, we have

$$\sum_{ik\in E(\dot G)} w_2^+(i,k)x_k - \sum_{ik\in E(\dot G)} w_2^-(i,k)x_k = \sum_{ik\in E(\dot G)} \big(w_2^+(i,k) - w_2^-(i,k)\big)x_k$$
$$\leq \sum_{ij\in E(\dot G)} \big|w_2^+(i,j) - w_2^-(i,j)\big|x_i.$$

For the second and the fourth term, we have

$$\sum_{ik\notin E(\dot G)} w_2^+(i,k)x_k - \sum_{ik\notin E(\dot G)} w_2^-(i,k)x_k \leq \sum_{ik\notin E(\dot G)} \big(w_2^+(i,k) + w_2^-(i,k)\big)x_i$$
$$= \sum_{ij\in E(\dot G)} \big(d_j - |N_i\cap N_j|\big)x_i.$$

Inserting the previous inequalities in (9), we get

$$\big(\nu_1^2 - d_i^{\pm}\nu_1\big)x_i \leq m_i^{\pm}x_i + n_ix_i,$$

which gives (8).

If $\dot G$ is regular with all edges being positive, then $d_i^{\pm} = d_i$, $m_i^{\pm} = n_i = d_i^2$, and so the right-hand side of (8) reduces to $2d_i$, and this is exactly the largest eigenvalue of the Laplacian matrix of a regular graph of degree d_i. ■

Here is a simple corollary.

Corollary 4.1. *Under the notation of Theorem* 7*, if $\dot G$ is triangle-free, then*

$$\nu_1 \leq \max\left\{\frac{1}{2}\left(d_i^{\pm} + \sqrt{{d_i^{\pm}}^2 + 4\sum_{ij\in E(\dot G)}\big(|d_j^{\pm}| + d_j\big)}\right) : 1\leq i\leq n\right\}.$$

Proof. If $\dot G$ is triangle-free, then $n_i = \sum_{ij\in E(\dot G)}\big(\big|d_j^{\pm}\big| + d_j\big)$ (as $|N_i\cap N_j| = 0$, for $ij\in E(\dot G)$), which leads to the result. ■

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