GD-SAT model and crossover line

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Abstract

In the last decade a lot of effort has been invested into both theoretical and experimental analysis of SAT phase transition. However, a deep theoretical understanding of this phenomenon is still lacking. It is still a very challenging problem to determine a relationship between crossover points for different SAT problems. This paper introduces one new class of randomly generated SAT problems, GD-SAT, and we experimentally show there is a phase transition for the problems in this class. On the basis of both analytical and experimental arguments we conjecture that there is a surprisingly simple, linear relationship between crossover points for problems in this class. This relationship is of both theoretical and practical importance.

Keywords: SAT problem, phase transition, NP completeness

1 Introduction

In recent years, phase transition for many NP-hard problems has been the subject of both theoretical and experimental consideration (some of the first were the influential papers by Cheesman *et. al.* and by Mitchell *et. al.* published in early 1990s [Cheeseman *et al*, 1991, Mitchell *et al*, 1992]). A prototypical example of such problems is propositional satisfiability problem — SAT (SAT is the problem of deciding if there is an truth assignment for which a given propositional formula is evaluated to true; it was shown by Cook that SAT is NP-complete problem [Cook, 1971]). This paper focuses on SAT problems in conjunctive normal form: (N, L)-SAT problem consists of L clauses over the set of N variables and their negations (in the rest of the paper, by SAT problem we mean a problem of this form). Many experiments (over problem sets with different additional syntactical constraints) suggest that there is a phase transition in SAT problems between satisfiability and unsatisfiability as the ratio L/N is increased. For most of SAT problem sets it is easy to show that the percentage of satisfiable formulae strictly decreases and converges to 0 as L increases. For different types of problem sets, it is conjectured that there is a value c_0 of L/N, which we call a *crossover point* (or a *phase transition point*) such that:

$$\lim_{N\to\infty} s(N,[cN]) = \begin{cases} 1, & \text{for } c < c_0 \\ 0, & \text{for } c > c_0 \end{cases}$$

where $s(N,L)$ is a *satisfiability function* that maps sets of propositional formulae (of L clauses over N variables) into the segment $[0,1]$ and corresponds to a percentage of satisfiable formulae. The value of the crossover point might be (and most often is) different for different types of problem sets. For a fixed problem set, the sequence of points L/N in which the satisfiability function is (approximately) equal p (where $0\% < p < 100\%$), converges to the crossover point as N increases; in most of the experiments, crossover points for different SAT problems is estimated (usually using $p = 50\%$) on the basis of this fact. Additionally, experimental results suggest that at the crossover point approximately the same percentage of formulae is satisfiable for large values of N (while that percentage depends on SAT model examined) [Larrabee & Tsuji, 1992, Gent & Walsh, 1994].

As yet, for none of SAT problem sets the crossover point has been theoretically computed nor even proved that it exists (with the only exception of 2-SAT problem). However, recent Friedgut's results [Friedgut, 1999] serve as a major step towards solving this problem: he proved that the transition region for k-SAT problems narrows as the number of variables increases (despite that, as Friedgut says, it is still feasible that, though there is a swift transition of the satisfiability function, the critical value does not converge to any given value).

Experimental results also suggest that in all SAT problems there is a typical easy-hard-easy pattern as the ratio L/N is increased. Indeed, for small values of L/N, problems are *under-constrained* and (relatively) easy for all propositional decision procedures because there are many satisfying assignments; for large values of L/N, problems are *over-constrained*, thus (relatively) easily shown to be unsatisfiable. Interestingly, the most difficult SAT problems for all decision procedures for propositional logic are those in the crossover region. All known decision procedures for propositional logic are of exponential worst-case complexity. Decision procedures most often considered in SAT experiments are Davis-Logemann-Loveland's procedure (often misattributed to Davis and Putnam), resolution based procedures and tableau based procedures. This paper mostly discusses satisfiability function and we are not much concerned by behavior of particular decision procedures.

There is a number of reports about different SAT models. For these models crossover points are determined (or approximated, on the basis of large number of randomly generated instances). However, there is a very small amount of literature on correspondence between crossover points for different SAT models. Even for some parametrized models (such as constant probability model) there are no discussion on crossover point as a function of the probability parameter (which determines the crossover curve), or it is not likely that this function might be represented as an elementary function. For $2+p$-SAT model, there are some results about a crossover curve [Monasson *et al*, 1996, Achlioptas *et al*, 1997]. Although the simple $1/(1-p)$ function fits crossover points for probability parameter p ranging from 0 to approximately 0.4, it seems there are no clear promises that the crossover curve can be described by some simple function for p ranging from 0 to 1. If there would be such a function, it would, obviously, also give the crossover point for 3-SAT exactly.

This paper introduces one new SAT model for generating random propositional formulae. This model behaves surprisingly regularly — its crossover curve made of crossover points for all values of p (we define the crossover curve in section 3.3) is linear (for a properly chosen argument) and, consequently, satisfiability functions for GD-SAT problems for different values of a probability parameter p can be simply scaled into a single function with an appropriately chosen parameter (instead of with generally used parameter L/N). These conjectures are made on the basis of some motivating theoretical analysis and on the basis of experiments reported in the following text. Taking these conjectures true, it follows that all GD-SAT problems (both from classes P and NP) for a properly chosen parameter have a (unique) crossover point at 1.00.

Overview of the paper. Section 2 discusses most often models used in random generation of SAT instances. Section 3 introduces GD-SAT model and some of its properties: experimental results that illustrate a phase transition for this model, experimental results concerning crossover curve and, finally, experimental results for GD-SAT model using an unifying, unique parameter (instead of using the parameter L/N) are presented. Section 4 discusses further work and section 5 draws some final conclusions. The appendix gives numerical results of the experiments performed.

2 Random SAT models

Most of SAT experiments are conducted in the following way: for some N and for L/N varied by some constant, generate randomly (large) number of formulae of SAT corpora of some fixed sort; for large samples of formulae, a percentage of satisfiable formulae approximates the satisfiability function $s(N, L)$ for that SAT model. Usually, it is not checked if some formula occurs more than once. The crossover point for that model is usually determined in the following way: for each N there is approximated a critical point at which there are 50% satisfiable formulae (actually, instead of 50%, it can be taken any percentage other than 0% and 100%); the sequence of these critical points converges to the crossover point as N increases. Alternatively, a crossover point (for a fixed model) can be estimated as a point at which percentage of satisfiable formulae is approximately constant for large values of N.

The following models are used most often (the first one is a fixed clause length model, while the remaining three are random clause length models):

Random k-SAT model (Fixed clause length model): For given values N and L, an instance of random k-SAT formula is produced by randomly generating L clauses of length k. Each clause is produced by randomly choosing k distinct variables from the set of N available variables, and negating each with probability 0.5 [Mitchell *et al*, 1992]. It is known that k-SAT is NP–complete for natural numbers k such that $k > 2$. There is a polynomial decision procedure for 2-SAT problem (i.e., 2-SAT $\in P$), but still there is a phase transition as for k-SAT problems for $k > 2$. It is proved that the crossover point for 2-SAT problems is 1 [Goerdt, 1992]. For random 3-SAT the phase transition occurs at $L/N \approx 4.25$ [Crawford & Auton, 1996]. For random 4-SAT the phase transition occurs at $L/N \approx 9.76$ [Gent & Walsh, 1994]. For large k, Kirkpatrick and Selman estimate the crossover points for k-SAT at $L/N = -1/\log_2(1 - 1/2^k)$ [Kirkpatrick & Selman, 1994]. It has been shown theoretically that the crossover point for 3-SAT is (if it exists) between 3.003 and 4.87 [Kamath *et al*, 1994]. Friedgut proved that the transition region for k-SAT problems narrows as the number of variables increases [Friedgut, 1999].

Constant probability model: In this model [Hooker & Fedjki, 1990], given N variables and L clauses, each clause is generated so that it contains each of $2N$ different literals with probability p. Some experiments use a variant of this model: if an empty clause or a unit clause is generated, it is discarded and another clause is generated in its place. Parameter p can be chosen such that $2Np = 3$ and then the mean clause length remains approximately constant as N varies [Gent & Walsh, 1994]. It is shown that there is a phase transition between satisfiability and unsatisfiability for constant probability model as L/N is varied and for $2Np = 3$, the crossover point is approximated as $L/N \approx 2.80$ [Gent & Walsh, 1994].

Random mixed SAT: In this model [Gent & Walsh, 1994], each clause is generated as in random k-SAT except that k (the length of clauses), is chosen randomly according to a finite probability distribution ϕ on integers. For instance, if $\phi(2) = 1/3$ and $\phi(4) = 2/3$, clauses of length 2 appear with the probability 1/3 and clauses of length 4 with the probability 2/3 (this problem is then called $2, 4, 4$-SAT). For random $2, 4, 4$-SAT, the phase transition occurs at $L/N \approx 2.74$ [Gent & Walsh, 1994].

$2 + p$-SAT **model** In this model [Monasson *et al*, 1996], a formula with L clauses has (approximately) $(1 - p)L$ clauses of the length 2 and pL clauses of the length 3.[1] Hence, a model smoothly interpolates between 2-SAT and 3-SAT

[1] This model is closely related to the random mixed SAT and can be considered as its special case.

model. Crossover points are approximated for different values between 0 and 1. For $p \leq 0.4$ it has been proved that the crossover point is at $L/N = 1/(1-p)$ [Achlioptas *et al*, 1997]. In addition, $2+p$-SAT behaves as 2-SAT for $p \leq 0.4$ and as 3-SAT for $p > 0.4$.

3 GD-SAT model

This section introduces a new sort of SAT problems, GD-SAT, which is based on geometric distribution on clause lengths. A phase transition detected for this model and report on an elegant relationship between crossover points for different GD-SAT problems are reported on here.

3.1 Generating clauses in GD-SAT model

Definitions of random clause lengths SAT problems typically include information on the distribution of clause lengths. For instance, the constant probability model has a limiting distribution on clause lengths determined by the Poisson distribution with parameter $2Np$ (adjusted for the omission of clauses of length 0 and 1), random mixed SAT has a finite discrete distribution on clause lengths etc. The SAT model based on geometric distribution of clause lengths is considered, and, hence, is denoted by GD-SAT. In this model, generating of clauses over the set of N variables, for the probability parameter p $(0 < p \leq 1)$, is specified by the stochastic context–free grammar given in table 1 (a stochastic context–free grammar is a context–free grammar with a stochastic component which attaches a probability to each of the production rules and controls its use). Clauses are generated independently of each other.

#	Rule	Probability
1.	$\langle clause \rangle := \langle literal \rangle \vee \langle literal \rangle$	p
2.	$\langle clause \rangle := \langle clause \rangle \vee \langle literal \rangle$	$1-p$
3.	$\langle literal \rangle := \langle variable \rangle \mid \neg \langle variable \rangle$	0.5
4.	$\langle variable \rangle := v_1 \mid v_2 \mid \ \ldots \ \mid v_N$	$1/N$

Table 1: Stochastic grammar for generating GD-SAT clauses

We point out that there is not performed a check whether some variable occurs more times in one clause, whether in some clause there are both a variable and its negation or whether there are multiple occurrences of some clause in a formula generated.

By the given stochastic grammar, only clauses of length equal or greater than 2 can be generated. Therefore, there is no need for discarding any of generated clauses and the original, geometric distribution on clause lengths is kept intact (which is not the case with the constant probability model). Lengths of clauses in the GD-SAT model have a geometric distribution; the probability of a clause of length k ($k = 2, 3, \ldots$) is $p(1-p)^{k-2}$. According to the properties of geometric distribution, the most probable clause length is 2 (with the probability p), while the expected clause length is $1 + 1/p$. For $p = 1$, GD-SAT model is exactly 2-SAT model (and, hence, it belongs to the class P). For any fixed p such that $p < 1$, GD-SAT is NP-complete. As p decreases, GD-SAT problems smoothly interpolate between 2-SAT and NP-complete GD-SAT problems. This makes GD-SAT model convenient for exploring a computational cost for directly related P and NP-complete problems (in a similar manner as in $2+p$-SAT model).

3.2 Phase transition for GD-SAT problems

We have performed a series of preliminary experiments[2] and the results show that there is a typical phase transition in GD-SAT model. Figure 1 shows data for $p = 1/2$ and for $N = 50$ based on 1000 generated and tested formulae at one L/N point, with parameter L/N increasing by step 0.1. There is also the typical easy-hard-easy pattern concerning the computational cost for Davis-Logemann-Loveland's procedure as the ratio L/N increases. The most difficult GD-SAT problems are those in the crossover region. The expected clause length in GD-SAT for $p = 0.5$ is 3, but the average difficulty of the generated instances is less than for 3-SAT instances; this is due to the clauses of the length 2 which make problems easier (a similar behaviour is reported for the constant probability model [Gent & Walsh, 1994]).

3.3 Critical 50% curves and crossover curve

For fixed values of p and N we denote by $cp(p, N)$ the critical L/N point with 50% satisfiable formulae. For a fixed N, small changes of p lead to close critical points and these critical 50% points (corresponding to different values of p) form a critical curve. It can be parametrized by p, or, as we will see, more conveniently by $1/p$ ($1/p$ is equal to the expected clause length minus one). On the other hand, for a fixed value p, when N increases, the sequence of these critical points converges to a crossover point for p and all these crossover points form a crossover curve c (which can be parametrized by $1/p$). For satisfiability

[2] In our experiments we use Davis-Logemann-Loveland's procedure [Davis *et al*, 1962] for checking satisfiability. In the split rule we choose the most occurring variable as a split variable. We use the number of split rules applied as a measure of computational cost; we count only the number of proper split rules applied (and we don't count the pure literal rule and the unit clause rule). The pure literal rule and the unit clause rule may be considered as particular cases of split rule, but it is the proper split rule which is in the root of the exponential nature of the procedure. Other measure of computational cost give analogous results.

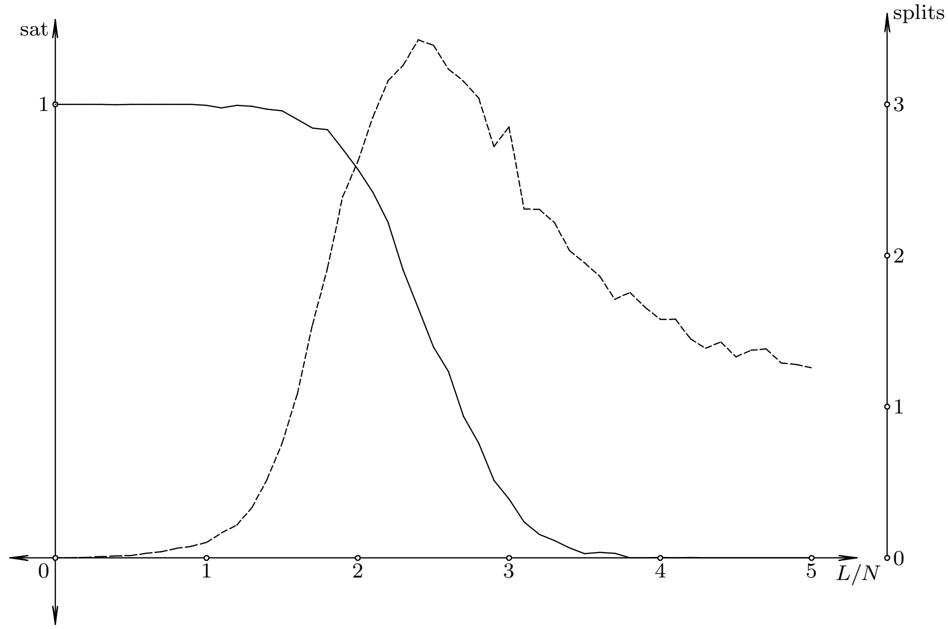

Figure 1: Satisfiability function (the solid line) and computational cost (the dashed line) for GD-SAT model for $p = 1/2$ as functions of ratio L/N ($N = 50$).

function for GD-SAT model $s(p, N, L)$ it holds:

$$\lim_{N\to\infty} s(p, N, L) = \begin{cases} 1, & L/N < c(1/p) \\ 0, & L/N > c(1/p) \end{cases}$$

In a quarter-plane $1/p \geq 1$, $L/N \geq 0$ each point corresponds to one set of GD-SAT problems. The crossover curve divides all of them into three classes: one with satisfiability function converging to 100% (when $N \to \infty$), one with satisfiability function converging to 0% and one made from the crossover curve itself. It is similar with critical curves for any fixed N: it divides points of $1/p \geq 1$, $L/N \geq 0$ quarter-plane into three classes: one with satisfiability function greater than 50%, one with satisfiability function less than 50% and one made from critical curve itself, i.e., one with satisfiability function equal 50%. If we know the exact positions of all critical curves and the exact position of crossover curve we could predict the behaviour of particular subclasses of GD-SAT problems (subclasses determined by the values p, N and L).

Preliminarily, in order to investigate properties of the crossover curve in GD-SAT model, we use Gent and Walsh's conjecture on location of crossover points in SAT models with random clause lengths [Gent & Walsh, 1994]. By this conjecture, if $\phi(k)$ is a distribution on clause lengths, c_ϕ crossover point for the model, and c_k $(k = 2, 3, \ldots)$ crossover points for k-SAT problem, it holds:

$$\frac{1}{c_\phi} = \sum_{k=2}^{\infty} \frac{\phi(k)}{c_k}$$

The above conjecture gives a good approximation for crossover point for different random clause lengths models [Gent & Walsh, 1994]. For large k, Kirkpatrick and Selman estimate the crossover points for k-SAT at $L/N = -1/\log_2(1-\frac{1}{2^k})$ [Kirkpatrick & Selman, 1994]. Thus, by these conjectures (using Kirkpatrick and Selman's estimate for all values k), since in GD-SAT $\phi(k) = p(1-p)^{k-2}$ $(k = 2, 3, \ldots)$, we can approximate the crossover points $c(1/p)$ in the following way:

$$\frac{1}{c(1/p)} \approx \sum_{k=2}^{\infty} -p(1-p)^{k-2} \log_2(1-\frac{1}{2^k})$$

If we approximate $\log_2(1-\frac{1}{2^k})$ by $-\frac{1}{2^k \ln 2}$ (which is, again, a good approximation for large k), then we have:

$$\frac{1}{c(1/p)} \approx \sum_{k=2}^{\infty} p(1-p)^{k-2} \frac{1}{2^k \ln 2} = \frac{p}{2^2 \ln 2} \sum_{k=2}^{\infty} \left(\frac{1-p}{2}\right)^{k-2}$$

$$= \frac{p}{4 \ln 2} \sum_{k=0}^{\infty} \left(\frac{1-p}{2}\right)^{k} = \frac{p}{4 \ln 2} \cdot \frac{1}{1-\frac{1-p}{2}} = \frac{p}{2 \ln 2 (1+p)}$$

which yields:

$$c(1/p) \approx 2 \ln 2 + 2 \ln 2/p$$

We also approximate Gent and Walsh's sums by computing the first 1000 summands and taking $c_2 = 1$, $c_3 = 4.25$, $c_4 = 9.76$ (and approximating other crossover points by Kirkpatrick and Selman's estimation). The obtained values for the crossover points are shown in figure 2. Both these results suggest that the crossover curve for GD-SAT model is surprisingly simple, i.e that it is linear curve (in parameter $1/p$). The next subsection will try to experimentally locate the crossover curve, by approximating crossover points on the basis of large samples of GD-SAT formula. The obtained results support the speculation that the crossover curve for GD-SAT is linear.

3.4 Experimentally approximating crossover curve

We approximate the values $cp(p, N)$ (critical L/N points with 50% satisfiable formulae) experimentally using the following simple approach based on binary search: we start with an interval $[a, b]$ (for the parameter L/N) large enough to include the critical point (say $[0, 20]$); then we approximate the satisfiability function at the point $[N \cdot (a+b)/2]/N$ (on the basis of 1000 generated and

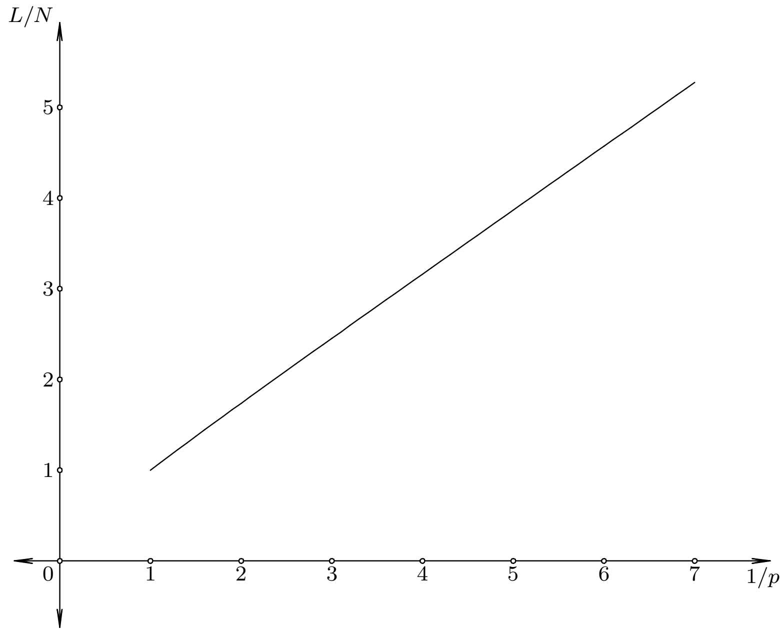

Figure 2: Approximation of the critical curve for GD-SAT model for and $1/p$ ranging from 1.0 to 5.0 on the basis of Gent and Walsh's conjecture

tested formulae); depending of its value (less than or equal to 50% or greater than 50%) we continue to search for the critical point in one of the (approximately) half-length intervals. We stop the search when the length of the current interval is equal to $1/N$ and then we approximate the critical 50% point (and computational cost) by simple, linear interpolation.

As said, our experiments are based on approximation of critical 50% points. Although some other methodology yield better estimates of crossover points, we use that approach as it gives information not only about crossover curve, but also about all critical 50% curves. On the basis of obtained experimental results, we can make predictions for behavior of GD-SAT formulae not only in limit (when $N \to \infty$) but for each particular N.

We know that the crossover curve passes through the point $(1, 1)$ (because the crossover point for 2-SAT problem is equal to 1) and we will try to determine it. We performed the following series of experiments: for a fixed N, for $1/p$ ranging from 1.0 to 5.0 by step 0.1, we experimentally approximate critical points with 50% satisfiable formulae. For fixed N, one value $1/p$ gives one value of L/N for which the percentage of satisfiable formulae is 50%. For fixed N, these points form one critical curve. Experimental results for $N = 50$ are shown in figure 3. Experimental results suggest that critical curve for $N = 50$ and for parameter $1/p$ is linear. We also measured computational cost in determined critical points

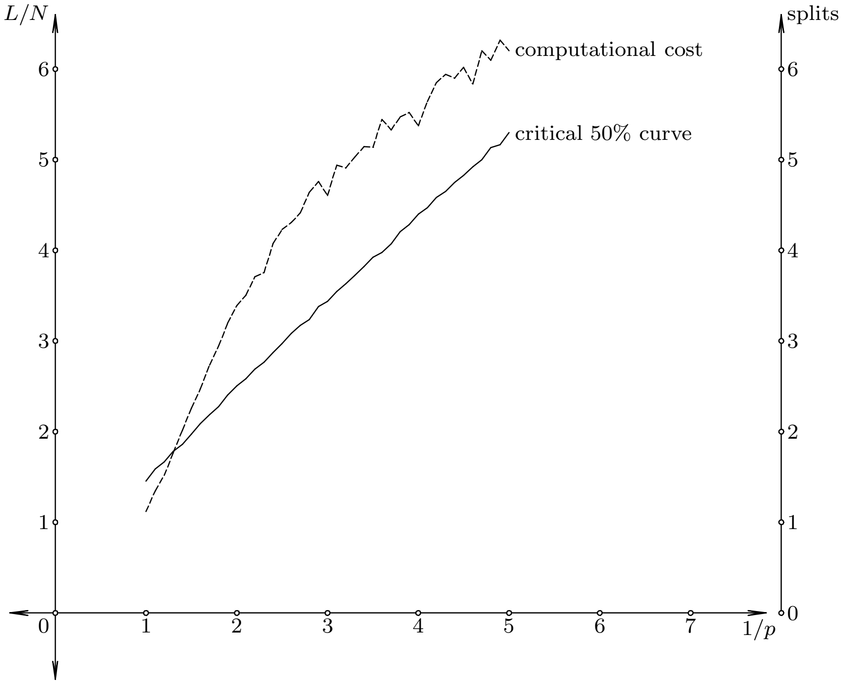

Figure 3: Critical curve (the solid line) and computational cost (the dashed line) for GD-SAT model for $N = 50$ and $1/p$ ranging from 1.0 to 5.0

and it is also shown in figure 3. It is likely that the computational cost function can be approximated by some elementary function (as speculated in section 4). However, this paper will be mainly concerned with satisfiability function (and not with computational cost). The same experiments were performed also for $N = 25$ and for $N = 100$. All obtained results are similar and suggest that all critical curves are lines. Critical 50% curves for $N = 25, 50, 100$ (and the approximation of the crossover curve by Gent and Walsh's conjecture) are shown in figure 4, computational cost is shown in figure 5, while numerical data are given in table 2 in Appendix. In figure 4 it seems that there are more noise for $N = 25$ than for $N = 50$ and $N = 100$. The probable explanation is in the way we approximate critical points, as for $N = 25$ we measure satisfiability function in points distanced $1/N = 0.04$, while for $N = 100$ in points distanced $1/N = 0.01$.

For a fixed N, we determined a line which is the least square fit (i.e., a line for which the sum of squares of residuals is the least possible) and we measured residuals for all points and for the fit given by this line. These residuals for $N = 50$ and for $1/p$ ranging from 1.0 to 15.0 by step 0.1 are shown[3] in figure 6

[3]We restricted our experiments to $N = 50$ and $1/p$ ranging from 1.0 to 15.0. Experiments

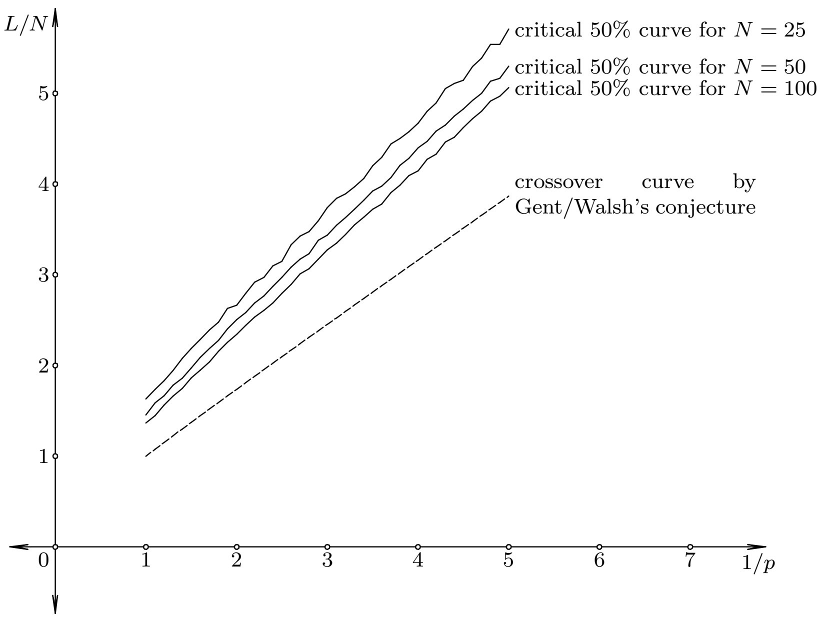

Figure 4: Critical curves with the parameter $1/p$ ranging from 1.0 to 5.0 for $N = 25$, $N = 50$ and $N = 100$ and the crossover curve on the basis of Gent and Walsh's conjecture

(the numerical data are given in Appendix, in table 3, the crossover curve and computational cost are shown in figure 10). Despite the noise, this closer look also suggest that the line is a good fit for the critical curve (as expected, the noise increases as $1/p$ increases). Similar results are obtained for other values for N as well. Since the critical curves are lines, their limit — the crossover curve is also a line. It is worth noticing that for all values N, critical points for values $1/p$ close to 1.0 were slightly less than points given by the least square fit. It would be interesting to further investigate this property (especially together with investigating computational cost and in the context of moving from P to NP-complete problems), but it goes beyond the scope of this paper; in addition, this property does not make a substantial impact on the location of critical curves.

Supported by the experimental results, assuming that all critical curves (curves for all values of N) are lines, we try to determine these lines. For each of them it is sufficient to determine just two points (that is why the parameter $1/p$ is much more convenient than the parameter p) and we do that for $1/p = 1.0$ and for $1/p = 5.0$. The good thing is that we know that the

with $N = 100$ and $1/p$ ranging from 1.0 to 15.0 (and using, say, 10000 formulae in each L/N point) would take weeks of CPU time.

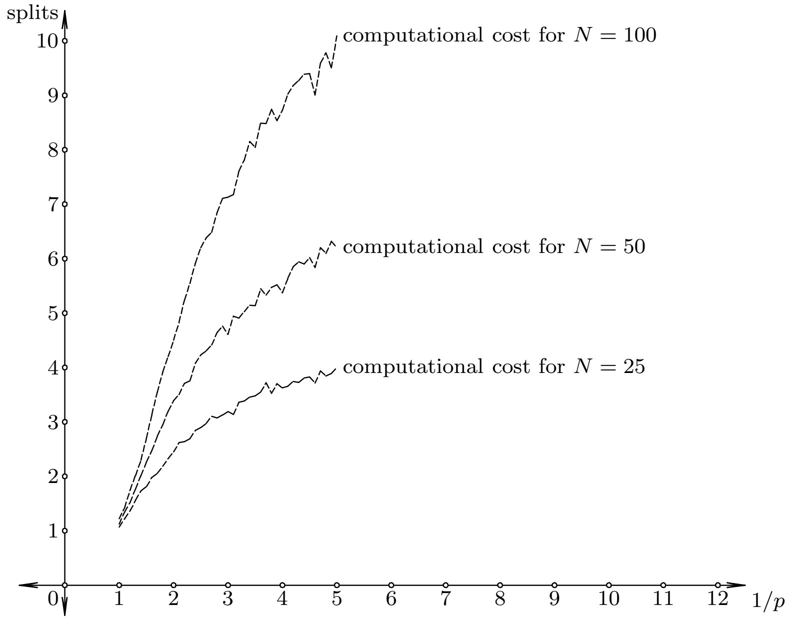

Figure 5: Computational cost with the parameter $1/p$ ranging from 1.0 to 5.0 for $N = 25$, $N = 50$ and $N = 100$

crossover point for 2-SAT problem is exactly 1. This implies that critical points for $1/p = 1$ converge to 1 (as N increases) and we know that the crossover line we are looking for passes through the point $(1, 1)$. That is why (apart from very consuming experiments) we use this approach instead of looking for least square fit lines.

The following series of experiments were performed: we approximated the critical 50% satisfiability points for $1/p = 1$ and $1/p = 5.0$ and for N ranging from 10 to 200 by step 10 (and in the way described in the above text). For each N, we determined the line $\alpha(N)(1/p) + \beta(N)$. The crossover curve is the line determined by limiting values of $\alpha(N)$ and $\beta(N)$ (for $N \to \infty$).

Figure 7 shows critical 50% points for $1/p = 1.0$ and $1/p = 5.0$ and for N ranging from 10 to 200. Notice the very slow converging of critical 50% points for $1/p = 1.0$ to the value 1 (it is known that $c(1.0) = 1.0$). Figure 8 shows coefficients $\alpha(N)$ and $\beta(N)$ of the lines $\alpha(N)(1/p) + \beta(N)$ determined by the above points (numerical data are given in table 4 in Appendix). Let $\alpha = \lim_{n\to\infty} \alpha(N)$ and $\beta = \lim_{n\to\infty} \beta(N)$. Thus, $c(1/p) = \alpha/p + \beta$. Since $c(1) = 1$, it holds $\alpha(N) + \beta(N) \to 1$ for $N \to \infty$ and $\alpha \cdot 1.0 + \beta = 1.0$. Therefore, $\beta = 1 - \alpha$. Results shown in figure 8 suggest that values $\alpha(N)$ are relatively stable and close to 0.9 and we conjecture that $\alpha \approx 0.9$ (and

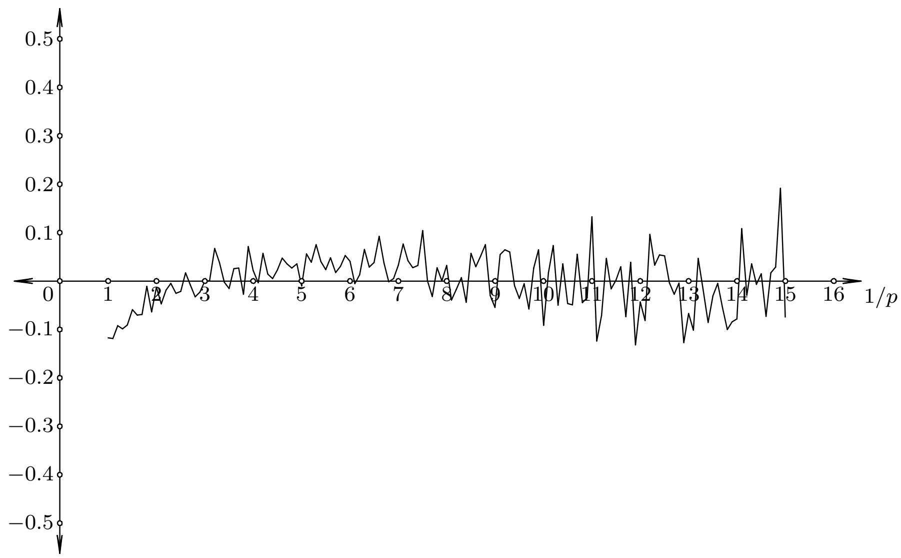

Figure 6: Residuals for critical points for $1/p$ ranging from 1.0 to 15.0, for $N = 50$ and for the least square fit

consequently $\beta \approx 0.1$). Thus, we conjecture

$$c(1/p) = \alpha/p + 1 - \alpha \approx 0.9/p + 0.1$$

Note that we haven't just approximated the crossover curve, but, by the values $\alpha(N)$ and $\beta(N)$ we can also approximate the critical 50% curves for different N. In practical problems that kind of information can be much more useful, i.e., the prediction of the satisfiability function is most often needed for some fixed value of N and not for the limiting case.

The crossover point for GD-SAT problems for $1/p = 2.0$ can be approximated on the basis of the fact that at the crossover point, satisfiability function (as a function of N) is approximately constant (for large values of N); at a point less than the crossover point, satisfiability function converges to 0 and at a point greater than the crossover point satisfiability function converges to 1. On the basis of the experiments (going up to $N = 2000$) it was shown that the satisfiability function slightly increases at $L/N = 1.9$ and slightly decreases at $L/N = 2.0$ [Janičić *et al*, 2000]. Thus, $c(2.0)$ is between 1.9 and 2.0, and taking $c(1/p) = \alpha/p+1-\alpha$, it follows $0.9 < \alpha < 1$ and $c(1/p) = \alpha/p+1-\alpha \approx 0.9/p+0.1$.

Note that Gent and Walsh's conjecture on location of crossover points (which support the conjecture of linearity of the crossover curve in GD-SAT model) gives $c(2.0) \approx 1.738$ (with $c_2 = 1$, $c_3 = 4.25$, $c_4 = 9.76$) while by our experimental data $c(2.0) \approx 1.9$. The probable explanation (apart from the imprecision of the experimental data) is that Kirkpatrick/Selman's estimation of the crossover

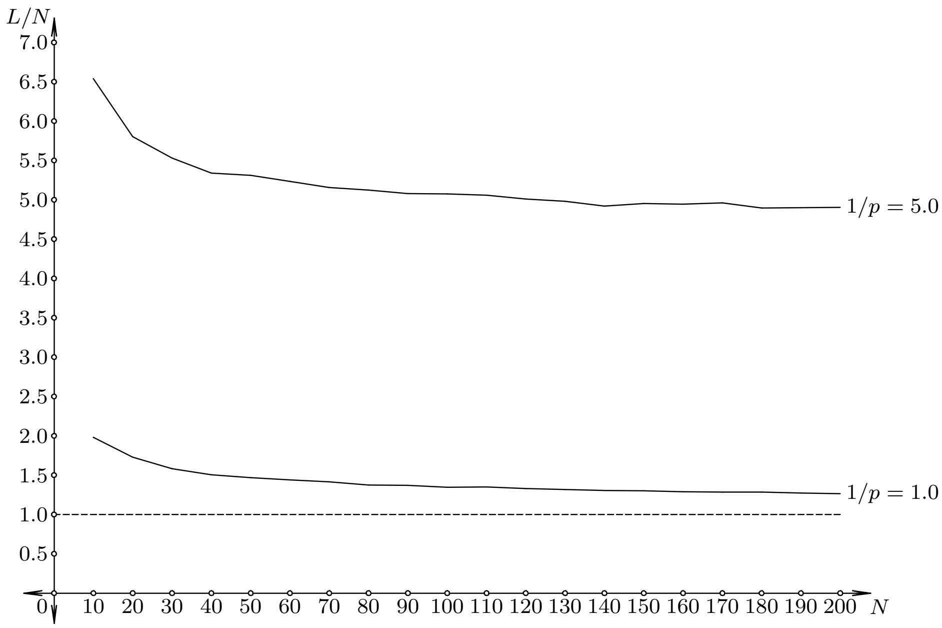

Figure 7: Critical 50% points for $1/p = 1.0$ and $1/p = 5.0$ and for N ranging from 10 to 200

points for k-SAT is not precise for small k or/and that Gent and Walsh's conjecture needs some specific refinement for some SAT models with random clause lengths.

3.5 Scaling satisfiability functions for GD-SAT problems

Assuming $c(1/p) = \alpha/p + 1 - \alpha \approx 0.9/p + 0.1$, this elegant relationship further suggests that there is a unique parameter for all GD-SAT problems — instead of the parameter L/N, we can consider the parameter $Lp/(N(0.9 + 0.1p))$ which gathers all (limiting) satisfiability functions for GD-SAT problems into one such function. Namely, denoting by $s(p, N, L)$ the satisfiability function for GD-SAT model (i.e., the percentage of satisfiable formulae for GD-SAT model for the probability parameter p, number of variables N and number of clauses L), we have:

$$\lim_{N\to\infty} s(p, N, [cN]) = \begin{cases} 1, & \text{for } c < 0.9/p + 0.1 \\ 0, & \text{for } c > 0.9/p + 0.1 \end{cases}$$

This relationship gathers together all GD-SAT limiting satisfiability functions (functions for all values of p) into one such function. It is interesting that satisfiability functions for both 2-SAT problem (i.e., GD-SAT for $1/p = 1$, which belongs to the class P) and for NP-complete GD-SAT problems are scaled into one

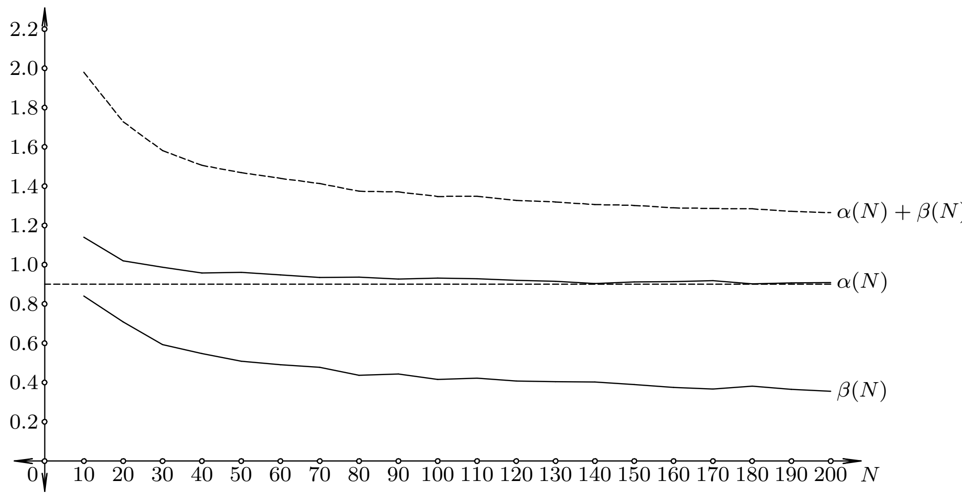

Figure 8: Coefficients $\alpha(N)$ and $\beta(N)$ of the lines $\alpha(N)\frac{1}{p}+\beta(N)$ (determined by the values of critical 50% points for $1/p = 1.0$ and $1/p = 5.0$ and for N ranging from 10 to 200)

such function. Figure 9 shows experimental results (numerical data are given in table 5 in Appendix) for GD-SAT problems with $Lp/(N(0.9+0.1p))$ parameter (for $N = 25$, $N = 50$ and $N = 100$): at each $Lp/(N(0.9+0.1p))$ point (ranging from 0 to 3 by step 0.1) we generated and tested 25 formulae for different values of p ($1/p$ is ranging from 1.0 to 5.0 by step 0.1), i.e., 1025 formulae at each $Lp/(N(0.9+0.1p))$ point. These experimental results illustrates the typical phase-transition behaviour and the typical easy-hard-easy pattern concerning the computational cost for Davis-Logemann-Loveland's procedure as the ratio $Lp/(N(0.9+0.1p))$ increases. As N increases, phase transition region narrows and critical 50% points converge to 1.0.[4]

Simple relationship between critical 50% points for GD-SAT problems enables us not only to scale all limiting GD-SAT satisfiability function into one limiting satisfiability function, but also gives us a possibility of estimating critical 50% points for different values of p and N only on the basis of critical points for two values of p. Moreover, all satisfiability functions (for different values of p) for any fixed N can be scaled (using parameters $\alpha(N)$ and $\beta(N)$) such that their

[4]For each SAT problem, at the crossover point approximately the same percentage of formulae is satisfiable for large values of N. However, by the given results, the satisfiability function for scaled GD-SAT problems has the same value for $N = 25$, $N = 50$ and $N = 100$ at the point $L/N \approx 0.9$. A possible explanation is threefold: (i) the parameter $1/p$ is restricted to 5.0; (ii) converging in GD-SAT model is slow and values 25, 50 and 100 for N are not large enough; (iii) the estimate $\alpha \approx 0.9$ can be further refined and lead to more precise scaling function.

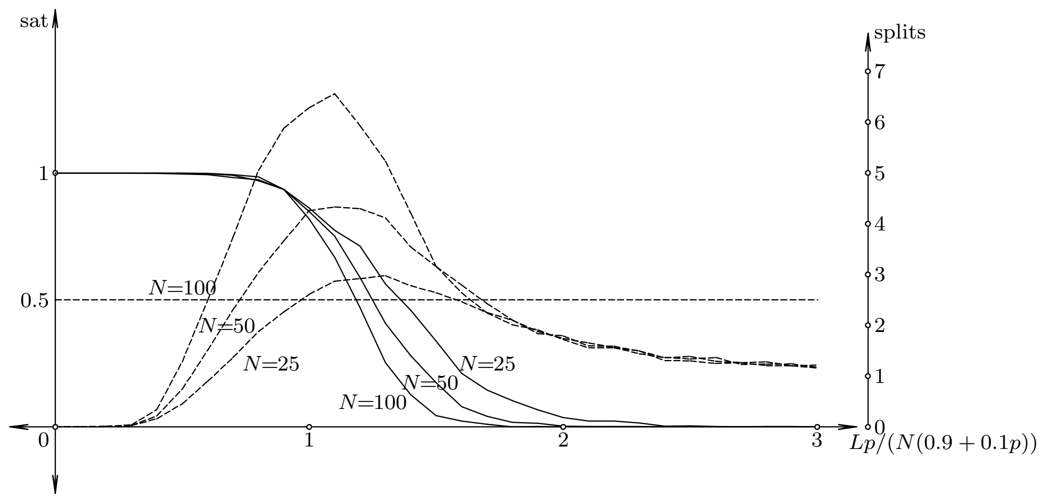

Figure 9: Satisfiability function (SAT) (solid lines) and computational cost (dashed lines) for GD-SAT problems for $1/p$ ranging from 1.0 to 5.0 and scaled on the parameter $Lp/(N(0.9 + 0.1p))$

50% critical points map into a unique such point.

4 Future work

In the future work we are planning to further refine experimental results reported (such as coefficients of critical lines and the limiting value α). In particular, we are planning to make more detailed investigation of locations of crossover points for values $1/p$ close to 1. Even if the satisfiability function for $1/p$ close to 1 slightly diverges from a linear curve, simple linear relationship enables determining all other crossover points (while there is no such elementary function for $2 + p$-SAT problem). Taking Gent and Walsh's conjecture true, the linear relationship between crossover points in GD-SAT model and facts such as $c_2 = 1$ can help in locating values c_k ($k = 3, 4, \ldots$). That direct link with crossover points for k-SAT problems is one of the most important features of GD-SAT model and we will try to explore it further.

Future work will also investigate the behaviour of computational cost (including mean and median values) for GD-SAT problems. We will try to investigate if there is a limit (for p) until which GD-SAT behaves as 2-SAT (by analogy with 2+p-SAT model). We will try to detect the relationship between values $1/p$, N, L and the corresponding computational cost. We speculate that the computational cost in critical 50% points (as a function of p and for fixed N) is of the form $\gamma(N)\sqrt{p}+\delta(N)$. For instance, we speculate that computational cost in critical 50% points (as a function of p) for $N = 50$ is approximately equal to $-10.8\sqrt{p} + 11.0$; figure 10 shows this *ad hoc* fit given by $-10.8\sqrt{p} + 11.0$ and the computational cost in critical 50% points for N and for $1/p$ ranging from 1.0 to 15.0 by step 0.1. The value $\delta(N)$ is a limiting mean number of branches for fixed N at critical 50% point as $1/p$ increases (as $1/p$ increases, expected length of clauses increases). These $\delta(N)$ values may serve as a measure of the efficiency of a specific decision procedures.

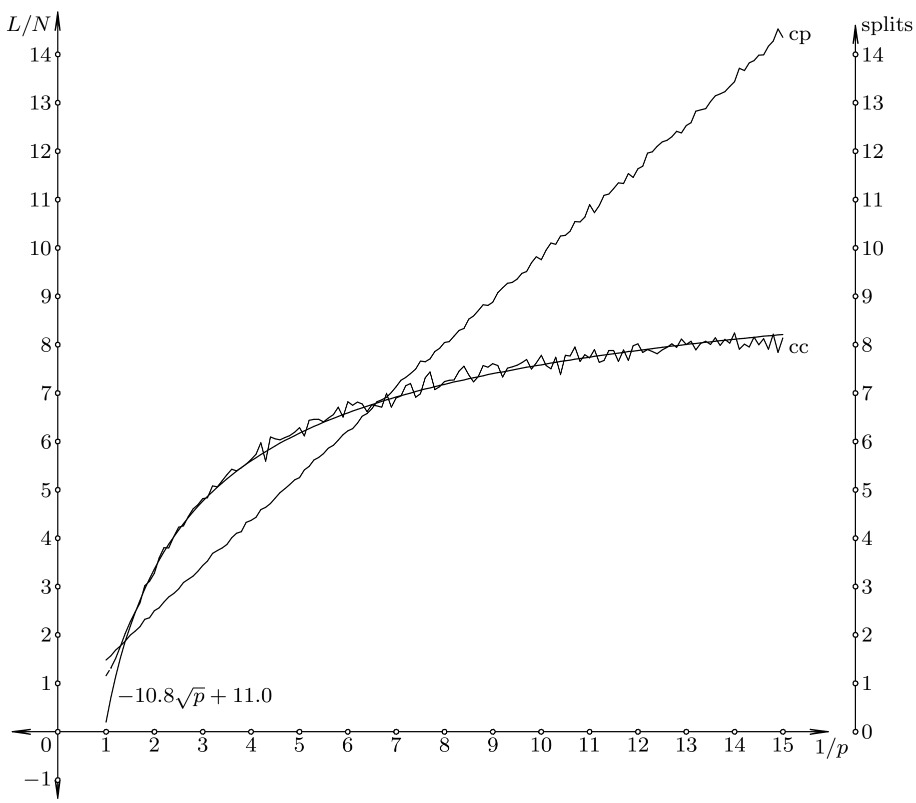

Figure 10: Critical curve (points CP), computational cost (points CC) for $N = 50$ and the function $-10.8\sqrt{p} + 11.0$ for $1/p$ ranging from 1.0 to 15.0 by step 0.1

Since GD-SAT problems directly link P and NP-complete problems, investigating computational cost could shed some new light not only to phenomenon of SAT phase transition, but also to relationships between classes of P and NP-complete problems.

In the future work we are also planning to try to theoretically explain the relationship between different GD-SAT problems reported in this paper and based on experimental results (and on Gent and Walsh's conjecture). In seems that GD-SAT model has some immanent elegant properties and we will try to explore possible relationship between it and some physical phenomena.

5 Conclusions

This paper has introduced one new random clause lengths SAT model — a model with geometry distribution on clause lengths (denoted GD-SAT model). Experiments were performed that showed the typical phase transition behaviour in GD-SAT model. In addition, on the basis of experimental results (and on the basis of some theoretical consideration, including Gent and Walsh's conjecture on crossover points in random clause lengths SAT models), we conjecture that the crossover point for GD-SAT problems with the probability parameter p is approximately equal to $0.9/p + 0.1$. Taking $1/p$ as parameter, all critical 50% curves and crossover curve are lines. This enables scaling of all GD-SAT limiting satisfiability functions into one such function: namely, taking a common parameter $Lp/(N(0.9 + 0.1p))$ for all GD-SAT problems (instead of the parameter L/N for each particular p) we get new sort of SAT problems with the crossover point equal 1.0. This is important both for theoretical and practical reasons. Namely, the GD-SAT model is a model in which some problems from both P and NP-complete classes have a common crossover point. We believe that this relationship could shed some new light both on SAT phase transition and on investigation of relations between classes P and NP. From the practical point of view, the above conjecture is important as it gives prediction of behaviour of a wide scale of propositional problems (corresponding different values of probability parameter in GD-SAT model). The key point of this paper is, thus, the linear crossover curve for GD-SAT model (while its more precise location is the subject of further experimental refinement).

In future work we are planning to further refine presented experimental results, to further investigate GD-SAT problems and their properties (such as computational cost) and we will also try to give theoretical explanations for the results reported in this paper.

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$1/p$	$N=25$		$N=50$		$N=100$	
	CP	CC	CP	CC	CP	CC
1.0	1.6295	1.0691	1.4553	1.1189	1.3650	1.2175
1.1	1.7378	1.2173	1.5877	1.3447	1.4476	1.4218
1.2	1.8327	1.3657	1.6650	1.5175	1.5627	1.7471
1.3	1.9436	1.5585	1.7834	1.7662	1.6610	2.0115
1.4	2.0771	1.7306	1.8589	2.0137	1.7458	2.2963
1.5	2.1890	1.8102	1.9731	2.2586	1.8628	2.6945
1.6	2.2853	1.9791	2.0897	2.4717	1.9471	3.1339
1.7	2.3920	2.0558	2.1896	2.7353	2.0409	3.5418
1.8	2.4778	2.1833	2.2768	2.9459	2.1574	3.9112
1.9	2.6284	2.3289	2.4017	3.1964	2.2550	4.1985
2.0	2.6628	2.4481	2.5069	3.3906	2.3426	4.5020
2.1	2.7940	2.6174	2.5827	3.5048	2.4413	4.8291
2.2	2.9200	2.6400	2.6886	3.7090	2.5324	5.2367
2.3	2.9719	2.6935	2.7667	3.7533	2.6088	5.5443
2.4	3.0985	2.8448	2.8727	4.0759	2.6893	5.9109
2.5	3.1486	2.8964	2.9715	4.2303	2.7954	6.2017
2.6	3.3314	2.9700	3.0838	4.3070	2.8947	6.3858
2.7	3.4257	3.1032	3.1712	4.4138	3.0108	6.4822
2.8	3.4744	3.0705	3.2340	4.6408	3.0676	6.8498
2.9	3.5966	3.1262	3.3819	4.7595	3.1720	7.1038
3.0	3.7400	3.1880	3.4363	4.6054	3.2751	7.1325
3.1	3.8426	3.1387	3.5460	4.9407	3.3481	7.1763
3.2	3.8922	3.3615	3.6302	4.9092	3.4443	7.6077
3.3	3.9700	3.3820	3.7235	5.0269	3.5505	7.8104
3.4	4.0623	3.4570	3.8200	5.1450	3.6343	8.1533
3.5	4.2029	3.4779	3.9231	5.1365	3.7223	8.0401
3.6	4.2954	3.5478	3.9764	5.4454	3.7784	8.4903
3.7	4.4414	3.7180	4.0714	5.3284	3.9062	8.4819
3.8	4.5011	3.5267	4.2067	5.4710	3.9867	8.7460
3.9	4.5760	3.7016	4.2817	5.5208	4.0911	8.5357
4.0	4.6697	3.6228	4.4010	5.3734	4.1467	8.7222
4.1	4.8022	3.6530	4.4711	5.6406	4.2722	9.0228
4.2	4.8945	3.7458	4.5819	5.8511	4.3292	9.1838
4.3	5.0518	3.7262	4.6516	5.9418	4.4636	9.2709
4.4	5.1098	3.8051	4.7507	5.9000	4.5176	9.3878
4.5	5.1448	3.8230	4.8267	6.0193	4.6208	9.3981
4.6	5.2960	3.7132	4.9218	5.8352	4.7204	9.0071
4.7	5.3886	3.9394	5.0000	6.2030	4.8036	9.5914
4.8	5.5378	3.8441	5.1327	6.0964	4.9132	9.7801
4.9	5.5408	3.8910	5.1656	6.3195	4.9674	9.5021
5.0	5.7083	3.9954	5.3000	6.2010	5.0634	10.0935

Table 2: Critical 50% points (CP) (in terms of L/N) and computational cost (CC) in these points for $1/p$ ranging from 1.0 to 5.0 and for $N = 25, 50, 100$

1/p	CP	CC	1/p	CP	CC	1/p	CP	CC
1.0	1.4811	1.1560	1.1	1.5717	1.3187	1.2	1.6894	1.5141
1.3	1.7747	1.7669	1.4	1.8739	2.0355	1.5	1.9977	2.2773
1.6	2.0782	2.4744	1.7	2.1706	2.6615	1.8	2.3204	3.0146
1.9	2.3592	3.1107	2.0	2.5018	3.2770	2.1	2.5593	3.5933
2.2	2.6795	3.8056	2.3	2.7850	3.7963	2.4	2.8564	4.0383
2.5	2.9514	4.2257	2.6	3.0820	4.2549	2.7	3.1483	4.4438
2.8	3.2155	4.6078	2.9	3.3172	4.6893	3.0	3.4318	4.8157
3.1	3.5236	4.8365	3.2	3.6821	5.0801	3.3	3.7446	5.0617
3.4	3.7959	5.1852	3.5	3.8743	5.3099	3.6	4.0071	5.4250
3.7	4.1000	5.3940	3.8	4.1371	5.4610	3.9	4.3276	5.5333
4.0	4.3704	5.6266	4.1	4.4363	5.7355	4.2	4.5884	5.9727
4.3	4.6375	5.5924	4.4	4.7195	6.0917	4.5	4.8295	6.0562
4.6	4.9456	6.0326	4.7	5.0254	6.0801	4.8	5.1079	6.1207
4.9	5.2086	6.1904	5.0	5.2547	6.2828	5.1	5.4120	6.1136
5.2	5.4868	6.4366	5.3	5.6150	6.4563	5.4	5.6716	6.4601
5.5	5.7467	6.4033	5.6	5.8624	6.4866	5.7	5.9240	6.5586
5.8	6.0279	6.7057	5.9	6.1425	6.5021	6.0	6.2218	6.8212
6.1	6.2671	6.7518	6.2	6.3775	6.8158	6.3	6.5217	6.7721
6.4	6.5766	6.6179	6.5	6.6776	6.7563	6.6	6.8235	6.7324
6.7	6.8600	6.7090	6.8	6.9129	6.9931	6.9	7.0114	6.7111
7.0	7.1306	6.9009	7.1	7.2661	6.9376	7.2	7.3229	7.1500
7.3	7.4000	7.2010	7.4	7.4964	6.9141	7.5	7.6604	6.9845
7.6	7.6482	7.3192	7.7	7.7073	7.4389	7.8	7.8589	7.0722
7.9	7.9233	7.1307	8.0	8.0467	7.2357	8.1	8.0665	7.2662
8.2	8.1800	7.2730	8.3	8.2961	7.4550	8.4	8.3368	7.5521
8.5	8.5300	7.3785	8.6	8.5937	7.2318	8.7	8.7071	7.3463
8.8	8.8226	7.5667	8.9	8.8094	7.5371	9.0	8.8766	7.6096
9.1	9.0773	7.5667	9.2	9.1790	7.3319	9.3	9.2661	7.5171
9.4	9.2877	7.5441	9.5	9.3533	7.5767	9.6	9.4756	7.6284
9.7	9.5150	7.6960	9.8	9.6912	7.4943	9.9	9.8205	7.6225
10.0	9.7561	7.7785	10.1	9.9560	7.5682	10.2	10.1039	7.5046
10.3	10.0727	7.7392	10.4	10.2494	7.3809	10.5	10.2595	7.7831
10.6	10.3482	7.7656	10.7	10.5446	7.9514	10.8	10.5364	7.6515
10.9	10.6380	7.7908	11.0	10.8967	7.7272	11.1	10.7317	7.8970
11.2	10.8763	7.6354	11.3	11.0855	7.6720	11.4	11.1144	7.8758
11.5	11.2238	7.8789	11.6	11.3436	7.6512	11.7	11.3321	7.8947
11.8	11.5364	7.6760	11.9	11.4570	7.9778	12.0	11.6372	8.0210
12.1	11.6908	7.8373	12.2	11.9600	7.9020	12.3	11.9884	7.8629
12.4	12.1014	7.8166	12.5	12.1908	7.8895	12.6	12.2270	7.9473
12.7	12.2945	8.0150	12.8	12.4100	7.9485	12.9	12.3786	8.1149
13.0	12.5300	8.0030	13.1	12.5877	8.0698	13.2	12.8275	7.8941
13.3	12.8561	8.0263	13.4	12.8786	8.0695	13.5	13.0249	8.0048
13.6	13.1423	8.1462	13.7	13.1840	7.9862	13.8	13.2300	8.1090
13.9	13.3376	8.0332	14.0	13.4357	8.2403	14.1	13.7136	7.9052
14.2	13.6680	8.0100	14.3	13.8241	7.9547	14.4	13.8735	8.1554
14.5	13.9871	8.0011	14.6	13.9911	8.1236	14.7	14.1722	7.9046
14.8	14.2760	8.2206	14.9	14.5306	7.8431	15.0	14.3560	8.1378

Table 3: Critical 50% points (CP) (in terms of L/N) and computational cost (CC) in these points for $1/p$ ranging from 1.0 to 15.0 and for $N = 50$

N	$1/p = 1.0$		$1/p = 5.0$		$\alpha(N)$	$\beta(N)$
	CP	CC	CP	CC		
10	1.9795	0.9781	6.5381	2.2214	1.1397	0.8398
20	1.7273	1.0478	5.8038	3.4200	1.0191	0.7081
30	1.5806	1.1095	5.5308	4.4512	0.9875	0.5931
40	1.5059	1.1366	5.3375	5.4055	0.9579	0.5480
50	1.4693	1.1391	5.3125	6.1845	0.9608	0.5084
60	1.4395	1.1432	5.2339	6.9621	0.9486	0.4908
70	1.4130	1.1697	5.1541	7.9686	0.9353	0.4778
80	1.3742	1.1412	5.1219	8.4973	0.9369	0.4372
90	1.3705	1.1848	5.0778	9.1450	0.9268	0.4437
100	1.3473	1.1992	5.0734	9.7966	0.9315	0.4158
110	1.3487	1.2025	5.0590	10.7244	0.9276	0.4212
120	1.3275	1.1776	5.0096	11.1383	0.9205	0.4070
130	1.3192	1.1935	4.9808	12.1270	0.9154	0.4038
140	1.3063	1.2224	4.9207	12.5966	0.9036	0.4026
150	1.3019	1.2357	4.9507	13.1864	0.9122	0.3897
160	1.2894	1.2318	4.9453	13.4125	0.9140	0.3754
170	1.2861	1.2455	4.9624	14.0437	0.9191	0.3670
180	1.2842	1.2781	4.8943	15.1457	0.9025	0.3817
190	1.2715	1.2504	4.8992	15.5268	0.9069	0.3646
200	1.2646	1.2747	4.9018	15.8684	0.9093	0.3554

Table 4: Critical 50% points (CP) (in terms of L/N), computational cost (CC) for $1/p = 1.0$ and $1/p = 5.0$ and coefficients $\alpha(N)$ and $\beta(N)$ of the lines $\alpha(N)\frac{1}{p} + \beta(N)$ (determined by the values of critical 50% points for $1/p = 1.0$ and $1/p = 5.0$) for N ranging from 10 to 200

$\frac{Lp}{N(0.9+0.1p)}$	$N = 25$		$N = 50$		$N = 100$	
	SAT	CC	SAT	CC	SAT	CC
0.0	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
0.1	1.0000	0.0020	1.0000	0.0000	1.0000	0.0000
0.2	0.9990	0.0049	1.0000	0.0068	1.0000	0.0137
0.3	1.0000	0.0273	1.0000	0.0283	1.0000	0.0400
0.4	0.9980	0.1610	1.0000	0.2156	1.0000	0.3405
0.5	0.9951	0.4546	0.9990	0.7444	0.9980	1.2839
0.6	0.9932	0.8956	0.9971	1.5083	0.9980	2.4917
0.7	0.9815	1.3659	0.9912	2.3044	0.9932	3.7288
0.8	0.9727	1.8761	0.9698	3.0420	0.9844	5.0488
0.9	0.9356	2.2634	0.9356	3.6615	0.9356	5.8810
1.0	0.8624	2.6088	0.8478	4.2576	0.8195	6.2839
1.1	0.7737	2.8663	0.7512	4.3288	0.6683	6.5590
1.2	0.7122	2.9180	0.5893	4.2966	0.4702	5.9307
1.3	0.5620	2.9776	0.4078	4.1132	0.2517	5.2312
1.4	0.4585	2.7776	0.2800	3.5405	0.1268	4.2068
1.5	0.3376	2.6410	0.1746	3.1580	0.0449	3.1600
1.6	0.2107	2.4624	0.0790	2.6410	0.0224	2.7902
1.7	0.1454	2.2527	0.0420	2.2420	0.0107	2.4283
1.8	0.1034	2.0995	0.0176	2.0107	0.0000	2.1063
1.9	0.0673	1.8780	0.0146	1.9102	0.0010	1.8371
2.0	0.0371	1.7395	0.0029	1.7210	0.0000	1.7932
2.1	0.0224	1.6527	0.0010	1.5561	0.0000	1.5941
2.2	0.0224	1.5610	0.0010	1.5600	0.0000	1.5873
2.3	0.0156	1.4937	0.0010	1.4410	0.0000	1.4927
2.4	0.0029	1.3600	0.0000	1.3659	0.0000	1.3054
2.5	0.0039	1.3863	0.0000	1.3385	0.0000	1.2995
2.6	0.0010	1.2917	0.0000	1.3659	0.0000	1.2498
2.7	0.0000	1.2644	0.0000	1.2302	0.0000	1.2624
2.8	0.0010	1.2810	0.0000	1.2380	0.0000	1.2049
2.9	0.0010	1.2010	0.0000	1.2400	0.0000	1.2029
3.0	0.0000	1.2156	0.0000	1.1737	0.0000	1.1600

Table 5: Satisfiability function (SAT) and computational cost for GD-SAT problems for $1/p$ ranging from 1 to 5 and scaled on the parameter $Lp/(N(0.9+0.1p))$