Одредити област конвергенције степеног реда

- $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}x^n$ $\qquad R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{|a_n|}}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}$

$|x|<\frac{1}{e}$ (C)

$|x|>\frac{1}{e}$ (D)

$x=-\frac{1}{e}$ $\qquad \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}\left(-\frac{1}{e}\right)^n=\sum_{n=1}^{\infty}(-1)^n\left(1+\frac{1}{n}\right)^{n^2}\frac{1}{e^n}$

$x=\frac{1}{e}$ $\qquad \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}\frac{1}{e^n}$ $\qquad \lim_{n\to\infty}a_n=e^{-\frac{1}{2}}\Rightarrow(D)$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2}\frac{1}{e^n}=e^{\lim_{n\to\infty}\ln\left(\left(1+\frac{1}{n}\right)^{n^2}\cdot\frac{1}{e^n}\right)}=e^{\lim_{n\to\infty}\left(n^2\ln\left(1+\frac{1}{n}\right)-n\right)}=\left|\frac{1}{n}=x\right|=e^{\lim_{x\to0}\left(\frac{1}{x^2}\ln(1+x)-\frac{1}{x}\right)}=$$

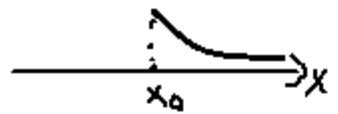
$$=e^{\lim_{x\to0}\frac{\ln(1+x)-x}{x^2}}\underset{\text{ЛП}}{\overset{\frac{0}{0}}{=}}e^{\lim_{x\to0}\frac{\frac{1}{1+x}-1}{2x}}=e^{\lim_{x\to0}\frac{-x}{2x(1+x)}}=e^{-\frac{1}{2}}\Rightarrow(D)$$

$\Rightarrow$ Одредити област $\quad x\in\left(-\frac{1}{e},\frac{1}{e}\right)$

- $\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}x^n$

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n\frac{\ln n}{n}}{(-1)^{n+1}\frac{\ln(n+1)}{n+1}}\right|=\lim_{n\to\infty}\frac{(n+1)\ln n}{n\ln(n+1)}=\lim_{n\to\infty}(1+\cancel{\tfrac{1}{n}}^{\,0})\cdot\lim_{n\to\infty}\frac{\ln n}{\ln(n(1+\frac{1}{n}))}=\lim_{n\to\infty}\frac{\ln n}{\ln n+\ln(1+\frac{1}{n})}=\lim_{n\to\infty}\frac{1}{1+\underbrace{\frac{\ln(1+\frac{1}{n})}{\ln n}}_{\to 0}}=1$$

$x=-1$ $\quad\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}(-1)^n=\sum_{n=2}^{\infty}\frac{\ln n}{n}$ $\qquad a_n=\frac{\ln n}{n}$ $\quad\lim\limits_{n\to\infty}a_n=$ $\qquad f(x)=\frac{\ln x}{x}$ $\quad\lim\limits_{x\to\infty}f(x)\overset{\frac{\infty}{\infty}}{\underset{\text{Л.П.}}{=}}\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{1}=+0\Rightarrow$



$\Rightarrow f(x)\searrow$ почев од $x_0\Rightarrow\lim\limits_{n\to\infty}a_n=0\ \land\ a_n\searrow$

$$\int_2^{\infty}\frac{\ln x}{x}dx=\lim_{t\to\infty}\int_1^{t}\frac{\ln x}{x}dx=\left|\begin{matrix}\ln x=z\\ \frac{dx}{x}=dz\end{matrix}\right|=\lim_{t\to\infty}\int_{\ln 2}^{\ln t}z\,dz=\lim_{t\to\infty}\frac{z^2}{2}\Big|_{\ln 2}^{\ln t}=\ldots=\infty\qquad(D)$$

$x=1$ $\quad\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}$ $\quad a_n=\frac{\ln n}{n}$ $\quad\Rightarrow(C)$ јер $a_n\searrow\to 0$

$\Rightarrow$ Одредити област $x\in(-1,1]$

- $\sum_{n=2}^{\infty} \frac{n^n}{(n!)^2} x^n$

- $\sum_{n=2}^{\infty} \frac{1}{n+a^n} x^n$

- $a>0$ $\quad \sum_{n=1}^{\infty} a^{\sqrt{n}} x^n$ $\quad a_n = a^{\sqrt{n}}$ $\quad R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a^{\sqrt{n}}|}} = \lim_{n\to\infty} \frac{1}{a^{\frac{1}{\sqrt{n}}}} = 1$

$x=-1$ $\quad \sum_{n=1}^{\infty} (-1)^n \cdot a^{\sqrt{n}}$ $\quad a_n = a^{\sqrt{n}}$ $\quad \lim_{n\to\infty} a_n = \left\{ \begin{array}{ll} 0, & a\in(0,1) \\ 1, & a=1 \\ \infty, & a\in[1,\infty) \end{array} \right\}$ (D) $\quad f(x)=a^x \downarrow$ за $a\in(0,1) \Rightarrow a_n \downarrow \Rightarrow$ (C)

$x=1$ $\quad \sum_{n=1}^{\infty} a^{\sqrt{n}}$ $\quad a_n \downarrow \to 0$ за $a\in(0,1)$

$$\int_1^\infty a^{\sqrt{x}}dx = \lim_{t\to\infty}\int_1^t a^{\sqrt{x}}dx = \left| \begin{array}{l} \sqrt{x}=z \\ x=z^2 \\ dx=2zdz \end{array} \right| = 2\lim_{t\to\infty}\int_1^{\sqrt{t}} a^z \cdot z\,dz = \left| \begin{array}{ll} u=z & dv=a^z dz \\ du=dz & v=\frac{a^z}{\ln a} \end{array} \right| = 2\lim_{t\to\infty}\left(\frac{a^z\cdot z}{\ln a}\Big|_1^{\sqrt{t}} - \frac{1}{\ln a}\frac{a^z}{\ln a}\Big|_1^{\sqrt{t}} \right) = 2\lim_{t\to\infty} \frac{a^{\sqrt{t}}}{\ln^2 a}(\sqrt{t}\ln a - 1) - \frac{a}{\ln a} + \frac{a}{\ln^2 a} =$$

$$= \frac{2}{\ln^2 a}\lim_{t\to\infty}\frac{\sqrt{t}\ln a - 1}{a^{-\sqrt{t}}} + \frac{a(1-\ln a)}{\ln^2 a} \overset{\frac{\infty}{\infty}}{\underset{\text{Л.П}}{=}} \frac{2}{\ln^2 a}\lim_{t\to\infty}\cancel{\frac{\frac{1}{2\sqrt{t}}\ln a}{a^{-\sqrt{t}}\cdot\ln a\cdot\frac{-1}{2\sqrt{t}}}}^{\,0} + \frac{a(1-\ln a)}{\ln^2 a} = \frac{a(1-\ln a)}{\ln^2 a} \quad (C)$$

$a\in(0,1)$ $\quad x\in[-1,1]$

$a\geq 1$ $\quad x\in(-1,1)$

- $\sum\limits_{n=2}^{\infty} \left[\frac{3^n}{n(3n+1)}\right]^{\alpha} x^n,\ \alpha \in R$

- $\sum\limits_{n=2}^{\infty} (-1)^n \frac{(n^2+3n)^p}{2n+1} x^n,\ p \in R$

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|(-1)^n \frac{(n^2+3n)^p}{2n+1}\right|}} = \lim_{n\to\infty} \frac{\sqrt[n]{2n+1}}{\left(\sqrt[n]{n^2+3n}\right)^p} = \lim_{n\to\infty} \frac{\underbrace{\sqrt[n]{2}}_{\to 1} \cdot \underbrace{\sqrt[n]{n}}_{\to 1} \cdot \underbrace{\sqrt[n]{1+\frac{1}{2n}}}_{\to 1}}{\left(\underbrace{\sqrt[n]{n^2}}_{\to 1} \cdot \underbrace{\sqrt[n]{1+\frac{3}{n}}}_{\to 1}\right)^p} = 1$$

$x=-1 \quad \sum\limits_{n=2}^{\infty} \frac{(n^2+3n)^p}{2n+1} \qquad a_n = \frac{(n^2+3n)^p}{2n+1}$

$$\lim_{n\to\infty} \frac{\frac{(n^2+3n)^p}{2n+1}}{\frac{1}{n^{\alpha}}} = \frac{1}{2} \lim_{n\to\infty} \frac{\frac{n^{2p}}{n}}{\frac{1}{n^{\alpha}}} = \frac{1}{2}$$

за $\alpha = 1-2p \qquad 1-2p>1 \Rightarrow p<0$ (C)

$p \geq 0$ (D)

$x=1 \quad \sum\limits_{n=2}^{\infty} (-1)^n \frac{(n^2+3n)^p}{2n+1}$

$$\lim_{n\to\infty} \frac{(n^2+3n)^p}{2n+1} = \begin{cases} 0, & 1>2p \\ \frac{1}{2}, & 1=2p \\ \infty, & 1<2p \end{cases}$$

$p<\frac{1}{2}$; $\left.\begin{matrix}1=2p\\1<2p\end{matrix}\right\}$ (D) $p \geq \frac{1}{2}$

$\lim\limits_{x\to\infty} \frac{(x^2+3x)^p}{2x+1} = +0 \quad f(x) \searrow$ монот. в ОД $x_0 \Rightarrow$ (C) за $p<\frac{1}{2}$

$$\left\{\begin{array}{ll} p \in (-\infty, 0) & x \in [-1, 1] \\ p \in [0, \frac{1}{2}) & x \in (-1, 1] \\ p \in [\frac{1}{2}, \infty) & x \in (-1, 1) \end{array}\right.$$

Табличне суме

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \ , \ x \in (-1,1)$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x \ , \ x \in (-\infty, \infty)$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n = \ln(1+x) \ , \ x \in (-1,1]$$

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = -\ln(1-x), \ x \in [-1,1)$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} = \sin x \ , \ x \in (-\infty, \infty)$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n} = \cos x \ , \ x \in (-\infty, \infty)$$

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^{\alpha}, \ x \in (-1,1)$$

Одредити област конвергенције и суму степеног реда

- $\sum_{n=1}^{\infty} \frac{1}{n+3} x^n$

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\frac{1}{n+3}\right|}} = \lim_{n\to\infty} \sqrt[n]{n} \cdot \sqrt[n]{1+\frac{3}{n}} = 1$$

$x=-1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n+3}(-1)^n \quad a_n = \frac{1}{n+3} \quad \lim_{n\to\infty} a_n = 0 \quad n+3 \nearrow \to a_n \searrow \Rightarrow (C)$

$x=1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n+3} \quad a_n = \frac{1}{n+3} \quad \lim_{n\to\infty} \frac{\frac{1}{n+3}}{\frac{1}{n^\alpha}} = 1$ за $\alpha = 1$ (D)

$x \in [-1, 1)$

$$S(x) = \sum_{n=1}^{\infty} \frac{1}{n+3} x^{n+3-3} = \frac{1}{x^3} \sum_{n=1}^{\infty} \frac{1}{n+3} x^{n+3} = \frac{1}{x^3}\left(\sum_{n=-2}^{\infty} \frac{1}{n+3} x^{n+3} - \underbrace{x}_{n=-2} - \underbrace{\frac{x^2}{2}}_{n=-1} - \underbrace{\frac{x^3}{3}}_{n=0} \right) = \frac{1}{x^3}\left(-\ln(1-x) - x - \frac{x^2}{2} - \frac{x^3}{3}\right), \quad x \in [-1,0) \cup (0,1)$$

$S(0) = 0$

• $\sum_{n=2}^{\infty}(-1)^n\frac{1}{n(n-1)}x^n$ $\qquad R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{\left|(-1)^n\frac{1}{n(n-1)}\right|}}=\lim_{n\to\infty}\underbrace{\sqrt[n]{n^2}}_{\to 1}\,\underbrace{\sqrt[n]{1-\frac{1}{n}}}_{\to 1}=1$

$x=-1$ $\quad \sum_{n=2}^{\infty}\frac{1}{n(n-1)}$ $\quad a_n=\frac{1}{n(n-1)}$ $\quad \lim_{n\to\infty}\frac{\frac{1}{n(n-1)}}{\frac{1}{n^2}}=1$ за $\alpha=2$ (C)

$x=1$ $\quad \sum_{n=2}^{\infty}(-1)^n\frac{1}{n(n-1)}$ $\quad a_n=\frac{1}{n(n-1)}$ $\quad \lim_{n\to\infty}a_n=0$ $\quad n(n-1)\nearrow$ за $n>1\Rightarrow a_n\searrow$ (C)

$\Big\}\ x\in[-1,1]$

$S(x)=\sum_{n=2}^{\infty}(-1)^n\frac{1}{n(n-1)}x^n\ \Big/'\ \Rightarrow S'(x)=\sum_{n=2}^{\infty}(-1)^n\frac{\cancel{n}}{\cancel{n}(n-1)}x^{n-1}=\sum_{n=2}^{\infty}(-1)^n\frac{1}{n-1}x^{n-1}=\ln(1+x)$

$S(x)=\int\ln(x+1)\,dx=\left|\begin{matrix}u=\ln(x+1) & dv=dx\\ du=\frac{dx}{x+1} & v=x\end{matrix}\right|=x\ln(x+1)-\int\frac{x+1-1}{x+1}\,dx=x\ln(x+1)-x+\ln(x+1)+C$

$S(0)=0=0\cdot\ln 1-0+\ln 1+C\Rightarrow C=0\Rightarrow S(x)=x\ln(x+1)-x+\ln(x+1),\ x\in(-1,1]$

$S(-1)=\sum_{n=2}^{\infty}\frac{1-n+n}{n(n-1)}=\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)=1-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\cdots=1$

• $\sum_{n=1}^{\infty}(n+(-1)^n 3^{n-1})x^n = \sum_{n=1}^{\infty} n\,x^n + \sum_{n=1}^{\infty}(-1)^n 3^{n-1}x^n$

$$\left.\begin{array}{ll} R_1 = \lim_{n\to\infty}\dfrac{1}{\sqrt[n]{|n|}} = 1 & |x|<1 \\ R_2 = \lim_{n\to\infty}\dfrac{1}{\sqrt[n]{|(-1)^n 3^{n-1}|}} = \lim_{n\to\infty}\dfrac{\sqrt[n]{3}}{3} = \dfrac{1}{3} & |x|<\dfrac{1}{3}\end{array}\right\} |x|<\frac{1}{3}$$

$$\rightarrow \left.\begin{array}{l} x=-\frac{1}{3}\quad \sum_{n=1}^{\infty}(-1)^n 3^{n-1}\left(-\frac{1}{3}\right)^n = \frac{1}{3}\sum_{n=1}^{\infty} 1 \quad (D) \\ x=\frac{1}{3}\quad \sum_{n=1}^{\infty}(-1)^n 3^{n-1}\left(\frac{1}{3}\right)^n = \frac{1}{3}\sum_{n=1}^{\infty}(-1)^n\cdot 1 \quad (D)\end{array}\right\} x\in\left(-\frac{1}{3},\frac{1}{3}\right)$$

$$S(x) = S_1(x)+S_2(x) = \frac{x}{(1-x)^2}+\frac{1}{3}\sum_{n=1}^{\infty}(-3x)^n = \frac{x}{(1-x)^2}+\frac{1}{3}(-3x)\sum_{n=1}^{\infty}(-3x)^{n-1} = \frac{x}{(1-x)^2} - x\frac{1}{1-(-3x)} = \frac{x}{(1-x)^2}-\frac{x}{1+3x}\ ,\ x\in\left(-\frac{1}{3},\frac{1}{3}\right)$$

$$\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}\ /' \Rightarrow \sum_{n=\cancel{0}\,1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}\ /\cdot x \Rightarrow \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

• $\sum_{n=1}^{\infty}\frac{n^2+3}{n!}x^n$

$$R = \lim_{n\to\infty}\left|\frac{\frac{n^2+3}{n!}}{\frac{(n+1)^2+3}{(n+1)!}}\right| = \lim_{n\to\infty}\frac{(n^2+3)(n+1)\cancel{n!}}{(n^2+2n+4)\cdot\cancel{n!}} = \infty \Rightarrow x\in(-\infty,\infty)$$

$$S(x) = \sum_{n=1}^{\infty}\frac{n^2}{n!}x^n + 3\sum_{n=1}^{\infty}\frac{x^n}{n!} = \sum_{n=1}^{\infty}\frac{n}{(n-1)!}x^n + 3\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}-\frac{x^0}{0!}\right) = e^x(x+x^2)+3(e^x-1) = e^x(x^2+x+3)-1\ ,\ x\in\mathbb{R}$$

$$\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = e^x\ /\cdot x \Rightarrow \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} = xe^x\ /' \Rightarrow \sum_{n=1}^{\infty}\frac{n x^{n-1}}{(n-1)!} = e^x + xe^x\ /\cdot x \Rightarrow \sum_{n=1}^{\infty}\frac{n x^n}{(n-1)!} = e^x(x+x^2)$$

- $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{4^n}{n(2n+1)}x^{2n} = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{4^n}{n(2n+1)}(x^2)^n$

$$R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{\left|(-1)^{n-1}\frac{4^n}{n(2n+1)}\right|}}=\frac{1}{4}\lim_{n\to\infty}\underbrace{\sqrt[n]{n^2}}_{\to 1}\cdot\underbrace{\sqrt[n]{2}}_{\to 1}\cdot\underbrace{\sqrt[n]{1+\frac{1}{2n}}}_{\to 1}=\frac{1}{4}$$

$x^2=\frac{1}{4} \Rightarrow x=\pm\frac{1}{2}$ $\quad \sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{\cancel{4^n}}{n(2n+1)}\cancel{\left(\pm\frac{1}{2}\right)^{2n}}=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n(2n+1)}$ $\quad a_n=\frac{1}{n(2n+1)}$ $\quad \lim\limits_{n\to\infty}a_n=0$ и $(2n+1)\nearrow$ за $n>0 \Rightarrow a_n\searrow \Rightarrow (C)$

$x\in\left[-\frac{1}{2},\frac{1}{2}\right]$

$$S(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4^n}{n(2n+1)}x^{2n}\ /\cdot x \Rightarrow xS(x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4^n}{n(2n+1)}x^{2n+1}\ /' \Rightarrow (xS(x))'=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4^n}{n\cancel{(2n+1)}}\cancel{(2n+1)}x^{2n}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4x^2)^n}{n}=\ln(1+4x^2)$$

$$xS(x)=\int\ln(1+4x^2)\,dx=\left|\begin{matrix}u=\ln(1+4x^2) & dv=dx\\ du=\frac{8x}{1+4x^2}dx & v=x\end{matrix}\right|=x\ln(1+4x^2)-2\int\frac{4x^2+1-1}{1+\underbrace{4x^2}_{(2x)^2}}dx=x\ln(1+4x^2)-2x+2\cdot\frac{1}{2}\operatorname{arctg}2x+C$$

$$xS(x)=x\ln(1+4x^2)-2x+\operatorname{arctg}2x+C\ \overset{x=0}{\Longrightarrow}\ 0=C$$

$$S(x)=\ln(1+4x^2)-2+\frac{1}{x}\operatorname{arctg}2x\ ,\quad x\in\left[-\frac{1}{2},0\right)\cup\left(0,\frac{1}{2}\right]\qquad S(0)=0$$

- $\sum_{n=2}^{\infty} n^2(x-2)^{n-2}$ а затим израчунати $\sum_{n=0}^{\infty} \frac{n^2}{4^n}$

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|n^2|}} = 1$$

$x-2=-1 \Rightarrow x=1 \qquad \sum_{n=2}^{\infty} n^2(-1)^{n-2} \qquad a_n = n^2 \quad \lim_{n\to\infty} a_n = \infty \ (D)$

$x-2 = 1 \Rightarrow x=3 \qquad \sum_{n=2}^{\infty} n^2 \qquad a_n = n^2 \quad \lim_{n\to\infty} a_n = \infty \ (D)$

$x \in (1,3)$

$$\sum_{n=0}^{\infty}(x-2)^n = \frac{1}{1-(x-2)} = \frac{1}{3-x} \ /' \Rightarrow \sum_{n=0}^{\infty} n(x-2)^{n-1} = \frac{1}{(3-x)^2} \ / \cdot (x-2)$$

$$\sum_{n=0}^{\infty} n(x-2)^n = \frac{x-2}{(3-x)^2} \ /' \Rightarrow \sum_{n=0}^{\infty} n^2(x-2)^{n-1} = \frac{(3-x)^2+(x-2)\cdot 2(3-x)}{(3-x)^4} = \frac{x-1}{(3-x)^3} \ / : (x-2) \Rightarrow \sum_{n=0}^{\infty} n^2(x-2)^{n-2} = \frac{x-1}{(x-2)(3-x)^3} \Rightarrow$$

$$0 + 1^2(x-2)^{-1} + \sum_{n=2}^{\infty} n^2(x-2)^{n-2} = \frac{x-1}{(x-2)(3-x)^3} \Rightarrow S(x) = \frac{x-1}{(x-2)(3-x)^3} - \frac{1}{x-2}, \ x \in (1,2)\cup(2,3) \ ; S(2) = 4$$

$$\sum_{n=0}^{\infty} \frac{n^2}{4^n}$$

- $\sum_{n=1}^{\infty} \frac{1}{(n+1)2^{n+1}}(x-3)^n$ а затим израчунати $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$