① Нека је $f(x)=\begin{cases} c(1-x)^2, & x\in[0,1] \\ 0, & x\notin[0,1] \end{cases}$

а) Одредити $c\in\mathbb{R}$ тако да је f густина расподеле неке случајне величине X

б) Одредити функцију расподеле те случајне величине X.

в) Израчунати $P\{X>\frac{1}{3}\}$.

г) Одредити $E(X)$ и $D(X)$.

а) $f\geq 0 \Rightarrow c\geq 0$

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow c\cdot\int_0^1 (1-x)^2 dx = 1$$

$$c\cdot \frac{(1-x)^3}{-3}\Big|_0^1 = 1$$

$$c\cdot\left(0+\frac{1}{3}\right)=1 \Rightarrow \boxed{c=3}$$

$\int_0^1 f(x)dx = 1$, јер

$$1=\int_{-\infty}^{\infty} f(x)dx = \underbrace{\int_{-\infty}^{0} f(x)dx}_{0} + \int_0^1 f(x)dx + \underbrace{\int_1^{+\infty} f(x)dx}_{0}$$

б) $$F_X(x)=\int_{-\infty}^{x} f(t)dt = \begin{cases} 0, & x\leq 0 \\ 1-(1-x)^3, & x\in(0,1] \\ 1, & x>1 \end{cases}$$

јер:

За $x\leq 0$: $F_X(x)=0$ јер је $f(x)=0$ за $x\leq 0$

За $x\in(0,1]$: $F_X(x)=\int_{-\infty}^{x} f(t)dt = \int_0^x 3(1-t)^2 dt = 3\cdot\frac{(1-t)^3}{-3}\Big|_0^x = -\left((1-x)^3-1\right)=1-(1-x)^3$

За $x>1$: $F_X(x)=\int_{-\infty}^{x} f(t)dt = \underbrace{\int_{-\infty}^{0} f(t)dt}_{0} + \underbrace{\int_0^1 f(t)dt}_{1} + \underbrace{\int_1^x f(t)dt}_{0} = 1$

в) $$P\{X>\tfrac{1}{3}\}=1-P\{X\leq \tfrac{1}{3}\} = 1-F_X\left(\tfrac{1}{3}\right) = 1-\left(1-\left(1-\tfrac{1}{3}\right)^3\right)=\left(\tfrac{2}{3}\right)^3=\tfrac{8}{27}$$

г) $$E(X)=\int_{-\infty}^{\infty} x f(x)dx = \int_0^1 3x(1-x)^2dx = 3\cdot\int_0^1 (x-2x^2+x^3)dx = 3\cdot\left(\frac{x^2}{2}\Big|_0^1 - \frac{2x^3}{3}\Big|_0^1 + \frac{x^4}{4}\Big|_0^1\right)$$

$$=3\cdot\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)=\frac{6-8+3}{12}\cdot 3 = \frac{1}{4}$$

$D(X)=E(X^2)-E(X)^2$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 3x^2(1-x)^2dx = 3\cdot\int_0^1 (x^2-2x^3+x^4)dx$$

$$=3\cdot\left(\frac{x^3}{3}\Big|_0^1 - \frac{2x^4}{4}\Big|_0^1 + \frac{x^5}{5}\Big|_0^1\right)=3\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)=\frac{10-15+6}{30}\cdot 3 = \frac{1}{10}$$

$$D(X)=\frac{1}{10}-\left(\frac{1}{4}\right)^2=\frac{1}{10}-\frac{1}{16}=\frac{8-5}{80}=\frac{3}{80}$$

② Случајна величина X има густину расподеле $f(x)=\begin{cases} a\cos x & , x\in[0,\frac{\pi}{2}] \\ 0 & , x\notin[0,\frac{\pi}{2}] \end{cases}$ где је a const.
Одредити $E(X)$ и $D(X)$.

$$f\geq 0 \Rightarrow a\geq 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{=0} + \int_{0}^{\frac{\pi}{2}} f(x)\,dx + \underbrace{\int_{\frac{\pi}{2}}^{+\infty} f(x)\,dx}_{=0} = 1$$

$$\int_0^{\frac{\pi}{2}} a\cos x\,dx = 1$$

$$a\cdot \sin x\Big|_0^{\frac{\pi}{2}} = 1$$

$$a\cdot 1 = 1 \quad \text{тј. } \boxed{a=1}$$

$$E(X)=\int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\frac{\pi}{2}} x\cos x\,dx = \begin{pmatrix} u=x & du=dx \\ dv=\cos x\,dx & v=\sin x \end{pmatrix}$$

$$= x\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\pi}{2}\sin\frac{\pi}{2} - 0 + \cos x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\frac{\pi}{2}} x^2\cos x\,dx = \begin{pmatrix} u=x^2 & du=2x\,dx \\ dv=\cos x\,dx & v=\sin x \end{pmatrix}$$

$$= x^2\sin x\Big|_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} x\sin x\,dx = \begin{pmatrix} u=x & du=dx \\ dv=\sin x\,dx & v=-\cos x \end{pmatrix}$$

$$= \left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2} - 0 - 2\left(-x\cos x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,dx\right)$$

$$= \frac{\pi^2}{4} - 2\cdot\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + 0 + \sin x\Big|_0^{\frac{\pi}{2}}\right) = \frac{\pi^2}{4} - 2\cdot 1 = \frac{\pi^2}{4} - 2$$

$$D(X) = E(X^2) - E(X)^2 = \frac{\pi^2}{4} - 2 - \left(\frac{\pi}{2} - 1\right)^2 = \frac{\pi^2}{4} - 2 - \left(\frac{\pi^2}{4} - \pi + 1\right) = \pi - 3$$

Пуасонова расподела

$X \in \mathcal{P}(\lambda)$ – X има Пуасонову расподелу $\mathcal{P}(\lambda)$ (X је случајна величина)

$\lambda > 0$

$X \in \mathcal{P}(\lambda)$ ако важи $P\{X=k\} = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad \forall k \in \mathbb{N}_0$

тј.
$$X : \begin{pmatrix} 0 & 1 & 2 & 3 & \dots \\ p_0 & p_1 & p_2 & p_3 & \dots \end{pmatrix}, \quad p_k = e^{-\lambda} \frac{\lambda^k}{k!}$$

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

3. Нека је $X \in \mathcal{P}(\lambda)$. Одредити $E(X)$ и $D(X)$.

$$E(X) = \sum_{k=0}^{\infty} k \cdot p_k = e^{-\lambda} \cdot \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \cdot \lambda \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$
$$= \lambda e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \cdot p_k = \sum_{k=1}^{\infty} k^2 p_k = e^{-\lambda} \cdot \sum_{k=1}^{\infty} k^2 \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \frac{\lambda^{k-1}}{(k-1)!} \cdot \lambda$$

$k = (k-1) + 1$

$$= e^{-\lambda} \cdot \lambda \cdot \left(\sum_{k=1}^{\infty} (k-1) \frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right)$$
$$= \lambda e^{-\lambda} \left(\sum_{k=2}^{\infty} \frac{\lambda^{k-1}}{(k-2)!} + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) = \lambda e^{-\lambda} \Big(\lambda \cdot \overbrace{\sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!}}^{e^{\lambda}} + e^{\lambda} \Big)$$

$$\sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} (\lambda e^{\lambda} + e^{\lambda}) = \lambda \cdot (\lambda + 1)$$

$$D(X) = E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Униформна расподела

Случајна величина X има униформну расподелу на интервалу $[a,b]$ у ознаци $X \in \mathcal{U}[a,b]$ ако важи:

$$f_X(x)=\begin{cases}\frac{1}{b-a}, & x\in[a,b]\\ 0, & x\notin[a,b]\end{cases}$$

$$f_X \geqslant 0 \quad \text{и} \quad \int_{-\infty}^{\infty} f_X(x)\,dx=\int_a^b \frac{1}{b-a}\,dx=\frac{1}{b-a}\cdot x\Big|_a^b=1$$

(4) Нека је $X \in \mathcal{U}[a,b]$.

а) Одредити функцију расподеле случајне величине X.

б) Одредити $E(X)$ и $D(X)$.

а)

$$F_X(x)=\int_{-\infty}^{x} f(t)\,dt \qquad f(t)=\begin{cases}\frac{1}{b-a}, & x\in[a,b]\\ 0, & x\notin[a,b]\end{cases}$$

за $x\leq a$: $F_X(x)=\int_{-\infty}^{x} \underbrace{f(t)}_{=0}\,dt=0$

за $x\in(a,b]$:

$$F_X(x)=\int_{-\infty}^{x} f(t)\,dt=\int_{-\infty}^{a} \underbrace{f(t)}_{=0}\,dt+\int_a^x f(t)\,dt=0+\int_a^x f(t)\,dt=\int_a^x \frac{1}{b-a}\,dt$$

$$=\frac{1}{b-a}\,t\Big|_a^x=\frac{x-a}{b-a}$$

за $x>b$:

$$F_X(x)=\int_{-\infty}^{x} f(t)\,dt=\int_{-\infty}^{a} \underbrace{f(t)}_{=0}\,dt+\underbrace{\int_a^b f(t)\,dt}_{=1}+\int_b^x \underbrace{f(t)}_{=0}\,dt=1$$

$$\Rightarrow F_X(x)=\begin{cases}0, & x\leq a\\ \frac{x-a}{b-a}, & x\in(a,b]\\ 1, & x>b\end{cases}$$

б)

$$E(X)=\int_{-\infty}^{\infty} x f(x)\,dx=\int_a^b \frac{x}{b-a}\,dx=\frac{1}{b-a}\cdot\frac{x^2}{2}\Big|_a^b=\frac{1}{b-a}\cdot\left(\frac{b^2}{2}-\frac{a^2}{2}\right)=\frac{b+a}{2}$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx=\int_a^b \frac{x^2}{b-a}\,dx=\frac{1}{b-a}\,\frac{x^3}{3}\Big|_a^b=\frac{1}{b-a}\cdot\frac{1}{3}\cdot(b^3-a^3)=\frac{a^2+ab+b^2}{3}$$

$$D(X)=E(X^2)-E(X)^2=\frac{a^2+ab+b^2}{3}-\frac{(a+b)^2}{4}=\frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}=\frac{a^2-2ab+b^2}{12}$$

$$D(X)=\frac{(a-b)^2}{12}$$

Биномна расподела

Случајна величина X има биномну расподелу $\mathcal{B}(n,p)$, у ознаци $X \in \mathcal{B}(n,p)$ где су $n \in \mathbb{N}$ и $p \in [0,1]$, ако важи:

$$P\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{за све } k \in \{0,1,\dots,n\}$$

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 & \dots & n \\ p_0 & p_1 & p_2 & p_3 & \dots & p_n \end{pmatrix}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Биномна формула

$$p_k = P\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p+1-p)^n = 1$$

5. Нека је $X \in \mathcal{B}(n,p)$ где је $n \in \mathbb{N}$ и $p \in [0,1]$.

а) Одредити $E(X)$ и $D(X)$.

б) Одредити расподелу случајне величине $Y = n - X$.

а)
$$E(X) = \sum_{k=0}^{n} k \cdot p_k = \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n}{k} \cdot \frac{(n-1)\cdots(n-k+1)}{(k-1)!} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n}{k} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= n \cdot p \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$$= n \cdot p \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} p^k \cdot (1-p)^{n-1-k} = np \cdot (p+1-p)^{n-1} = np$$

$$E(X^2) = \sum_{k=0}^{n} k^2 p_k = \sum_{k=1}^{n} k^2 \cdot \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k^2 \frac{n}{k} \binom{n-1}{k-1} \cdot p^k (1-p)^{n-k}$$

$$= n \cdot \sum_{k=1}^{n} k \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k} = np \cdot \sum_{k=1}^{n} (k-1+1) \binom{n-1}{k-1} p^{k-1} \cdot (1-p)^{n-k}$$

$$= np \cdot \left(\sum_{k=2}^{n} (k-1) \cdot \frac{n-1}{k-1} \cdot \binom{n-2}{k-2} p^{k-1} \cdot (1-p)^{n-k} + \underbrace{\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}}_{(p+1-p)^{n-1} = 1} \right)$$

$$= np \left((n-1) p \cdot \sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} (1-p)^{(n-2)-(k-2)} + 1 \right)$$

$$= np \cdot \left((n-1) p \cdot \sum_{k=0}^{n-2} \binom{n-2}{k} p^k \cdot (1-p)^{n-2-k} + 1 \right) = np \left((n-1) p \cdot (p+1-p)^{n-2} + 1 \right)$$

$$E(X^2) = np\left((n-1)p+1\right) = n(n-1)p^2 + np = n^2p^2 - np^2 + np$$

$$D(X) = n^2p^2 - np^2 + np - (np)^2 = np - np^2 = np(1-p)$$

б) $Y = n - X$

$$X : \begin{pmatrix} 0 & 1 & \cdots & n \\ p_0 & p_1 & \cdots & p_n \end{pmatrix}$$

$$P\{Y=k\} = P\{n-X=k\} = P\{X=n-k\} = p_{n-k}$$

$$Y : \begin{pmatrix} 0 & 1 & \cdots & n \\ p_n & p_{n-1} & \cdots & p_0 \end{pmatrix} \qquad p_{n-k} = \binom{n}{k} p^{n-k} (1-p)^{n-(n-k)}$$

$$P\{Y=k\} = \binom{n}{k} \cdot p^{n-k} \cdot (1-p)^k = \binom{n}{k} \cdot (1-p)^k \cdot p^{n-k}$$

$$k \in \{0,1,\dots,n\}$$

$$\Rightarrow Y \in B(n, 1-p).$$