⑥ домаћи :

$$\int x\sqrt{x^2+a^2}\,dx = \begin{pmatrix} \text{смена: } t = x^2+a^2 \\ dt = 2x\,dx \end{pmatrix} = \dots$$

$$\int x\sqrt{3x^2+2x+1}\,dx = \dots$$

① $a \neq 0$

згодно јер је $1+\cos 2t = 2\cos^2 t$, $1-\cos 2t = 2\sin^2 t$

$$I = \int\sqrt{\frac{a+x}{a-x}}\,dx = \begin{pmatrix} x = a\cos 2t \\ 2t\in(0,\pi),\ t\in(0,\frac{\pi}{2}) \\ dx = a\cdot(-\sin 2t)\cdot 2dt \end{pmatrix} = \int\sqrt{\frac{\cancel{a}(1+\cos 2t)}{\cancel{a}(1-\cos 2t)}}\cdot a(-\sin 2t)\cdot 2dt$$

$$= \int\sqrt{\frac{\cos^2 t\cdot \cancel{2}}{\sin^2 t\cdot \cancel{2}}}\,(-2a)\sin 2t\,dt = \int\frac{|\cos t|}{|\sin t|}(-2a)\cdot 2\sin t\cos t\,dt$$

$$= -4a\int\frac{\cos t}{\cancel{\sin t}}\cdot\cancel{\sin t}\cos t\,dt = -4a\cdot\int\cos^2 t\,dt = -4a\int\frac{1+\cos 2t}{2}\,dt$$

$|\cos t| = \cos t$, $|\sin t| = \sin t$ јер $t\in(0,\frac{\pi}{2})$

$$= -2a\left(t + \int\cos 2t\,dt\right)$$

$$= -2a\left(t + \frac{\sin 2t}{2}\right) + C$$

$\cos 2t = \frac{x}{a}$, $2t = \arccos\frac{x}{a} \in (0,\pi)$, $\sin 2t > 0$ за $2t\in(0,\pi)$ па је

$$\boxed{t = \frac{1}{2}\arccos\frac{x}{a}}$$

$$\sin 2t = \sqrt{1-\cos^2 2t} = \sqrt{1-\frac{x^2}{a^2}}$$

$$\boxed{\sin 2t = \frac{\sqrt{a^2-x^2}}{|a|}}$$

$$I = -2a\cdot\frac{1}{2}\arccos\frac{x}{a} - a\cdot\frac{\sqrt{a^2-x^2}}{|a|} + C$$

$$\boxed{I = -a\arccos\frac{x}{a} - \operatorname{sgn} a\cdot\sqrt{a^2-x^2} + C}$$

② $a \neq 0$, $I = \int x\sqrt{\frac{x}{2a-x}}\,dx = ?$ смена: $x = 2a\cos^2 t$ или $x = 2a\sin^2 t$

$$I = \int x\sqrt{\frac{x}{2a-x}}\,dx = \begin{pmatrix} x = 2a\sin^2 t,\ t\in(0,\frac{\pi}{2}) \\ dx = 2a\cdot 2\sin t\cos t\,dt \end{pmatrix} = \int 2a\sin^2 t\cdot\sqrt{\frac{2a\sin^2 t}{2a\cos^2 t}}\,4a\sin t\cos t\,dt$$

$$= 8a^2\int\sin^3 t\cos t\cdot\frac{|\sin t|}{|\cos t|}\,dt = 8a^2\int\sin^4 t\,dt$$

$\sin t > 0$, $\cos t > 0$ на $(0,\frac{\pi}{2})$

$$\sin^2 t = \frac{1-\cos 2t}{2},\quad \sin^4 t = \frac{1}{4}\left(1-2\cos 2t+\cos^2 2t\right) = \frac{1}{4}\left(1-2\cos 2t+\frac{1+\cos 4t}{2}\right)$$

$$\sin^4 t = \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t$$

$$\int\sin^4 t\,dt = \frac{3}{8}t - \frac{1}{2}\int\cos 2t\,dt + \frac{1}{8}\int\cos 4t\,dt = \frac{3}{8}t - \frac{1}{2}\,\frac{\sin 2t}{2} + \frac{1}{8}\,\frac{\sin 4t}{4} + C$$

$$\sin t = \sqrt{\frac{x}{2a}},\quad \cos t = \sqrt{1-\frac{x}{2a}} = \sqrt{\frac{2a-x}{2a}}\quad (\text{јер је } \cos t > 0)$$

$$\sin 2t = 2\sin t\cos t = 2\cdot\sqrt{\frac{x}{2a}}\cdot\sqrt{\frac{2a-x}{2a}} = \sqrt{\frac{x(2a-x)}{a^2}} = \frac{1}{|a|}\cdot\sqrt{x(2a-x)}$$

$$\cos 2t = \cos^2 t - \sin^2 t = 1 - \frac{x}{2a} - \frac{x}{2a} = 1 - \frac{x}{a} \qquad , t = \arcsin\sqrt{\frac{x}{2a}}$$

$$\sin 4t = 2\sin 2t\cos 2t = \frac{2}{|a|}\sqrt{x(2a-x)}\cdot\left(1-\frac{x}{a}\right)$$

$$I = 8a^2\cdot\left(\frac{3}{8}t - \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t\right) + C$$

$$I = 3a^2\cdot\arcsin\sqrt{\frac{x}{2a}} - 2a^2\cdot\frac{1}{|a|}\sqrt{x(2a-x)} + \frac{a^2}{4}\cdot\frac{2}{|a|}\sqrt{x(2a-x)}\left(1-\frac{x}{a}\right)$$

$$I = 3a^2\arcsin\sqrt{\frac{x}{2a}} - 2|a|\sqrt{x(2a-x)} + \frac{|a|}{2}\sqrt{x(2a-x)}\,\frac{a-x}{a} \qquad \left(\frac{|a|}{a} = \operatorname{sgn} a\right)$$

за $a \neq 0$

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③ $I_n = \int \sin^n t\,dt = ?$

$$I_1 = \int \sin t\,dt = -\cos t + C$$

$$I_2 = \int \sin^2 t\,dt = \int \frac{1-\cos 2t}{2}dt = \frac{1}{2}\left(t - \frac{\sin 2t}{2}\right) + C$$

$$I_n = \int \sin^n t\,dt = \int \sin^{n-1}t\cdot\sin t\,dt = \left(\begin{matrix} u = \sin^{n-1}t & du = (n-1)\sin^{n-2}t\cdot\cos t\,dt \\ dv = \sin t\,dt & v = -\cos t\end{matrix}\right)$$

$$I_n = -\cos t\cdot\sin^{n-1}t + (n-1)\int \sin^{n-2}t\cdot\underbrace{\cos^2 t}_{1-\sin^2 t}\,dt = -\cos t\sin^{n-1}t + (n-1)\Big(\underbrace{\int\sin^{n-2}t\,dt}_{I_{n-2}} - \underbrace{\int\sin^n t\,dt}_{I_n}\Big)$$

$$n.I_n = -\cos t\sin^{n-1}t + (n-1)I_{n-2}$$

$$\boxed{I_n = \frac{-\cos t\sin^{n-1}t}{n} + \frac{n-1}{n}I_{n-2}}$$

Од I_1 добијамо све I_{2k+1}

Од I_2 добијамо све I_{2k}

Нпр. $I_3 = \dfrac{-\cos t\sin^2 t}{3} + \dfrac{2}{3}I_1$

$I_4 = \dfrac{-\cos t\sin^3 t}{4} + \dfrac{3}{4}I_2$ итд.

домаћи: $J_n = \int\cos^n t\,dt$

Наћи рекурентну везу!

① $\int \frac{x\ln|x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx = I$

парцијална интеграција?

$$I = \int \frac{x\ln|x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx = \left(\begin{array}{ll} u = \ln|x+\sqrt{1+x^2}) & du = \frac{1}{x+\sqrt{1+x^2}}\left(1+\cancel{\frac{1}{2}}\frac{1}{\sqrt{1+x^2}}\cancel{2}x\right)dx \\ dv = \frac{x\,dx}{\sqrt{1+x^2}} & v = \int\frac{x\,dx}{\sqrt{1+x^2}} \overset{\text{1}}{=} \int\frac{dt}{2\sqrt{t}} = \cancel{\frac{1}{2}}\cdot\frac{t^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} = \sqrt{t} \end{array}\right.$$

$t = x^2+1$
$dt = 2x\,dx$

$$v = \sqrt{x^2+1}\quad , du = \frac{\overset{1}{\cancel{\sqrt{1+x^2}+x}}}{\cancel{(x+\sqrt{1+x^2})}\cdot\sqrt{1+x^2}}dx$$

$$I = \ln|x+\sqrt{1+x^2})\cdot\sqrt{x^2+1} - \int\frac{\sqrt{x^2+1}}{\sqrt{1+x^2}}dx = \ln|x+\sqrt{1+x^2})\sqrt{1+x^2} - x + C$$

②

$$\int(\arcsin x)^2dx = \left(\begin{array}{ll} u = (\arcsin x)^2 & du = 2\arcsin x\cdot\frac{dx}{\sqrt{1-x^2}} \\ dv = dx & v = x\end{array}\right)$$

$$= x(\arcsin x)^2 - 2\int\frac{\arcsin x\cdot x}{\sqrt{1-x^2}}dx = \left(\begin{array}{ll} u = \arcsin x & du = \frac{1}{\sqrt{1-x^2}}dx \\ dv = \frac{x\,dx}{\sqrt{1-x^2}} & v = \int\frac{x\,dx}{\sqrt{1-x^2}} = \left(\begin{array}{l} t = 1-x^2 \\ dt = -2x\,dx\end{array}\right.\end{array}\right.$$

$$= x(\arcsin x)^2 - 2\cdot\left(\arcsin x\cdot(-\sqrt{1-x^2}) - \int\frac{-\sqrt{1-x^2}}{\sqrt{1-x^2}}dx\right)$$

$$v = \int\frac{-dt}{2\sqrt{t}} = -\frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} = -\sqrt{t}$$

$$v = -\sqrt{1-x^2}$$

$$= x(\arcsin x)^2 + 2\arcsin x\sqrt{1-x^2} - 2\int dx$$

$$= x(\arcsin x)^2 + 2\arcsin x\sqrt{1-x^2} - 2x + C$$

③

$$I = \int x\arcsin(1-x)dx = \left(\begin{array}{ll} t = 1-x & dx = -dt \\ x = 1-t & \end{array}\right) = \int(1-t)\arcsin t(-dt)$$

$$= \int t\arcsin t\,dt - \int\arcsin t\,dt$$

$$\int\arcsin t\,dt = \left(\begin{array}{ll} u = \arcsin t & du = \frac{1}{\sqrt{1-t^2}}dt \\ dv = dt & v = t\end{array}\right) = t\arcsin t - \int\frac{t\,dt}{\sqrt{1-t^2}}$$

1 смена: $m = 1-t^2$
$dm = -2t\,dt$

$$= t\arcsin t - \int\frac{-dm}{2\cdot\sqrt{m}} = t\arcsin t + \cancel{\frac{1}{2}}\frac{m^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + C$$

$$= t\arcsin t + \sqrt{1-t^2} + C$$

$$\int t\arcsin t\,dt = \left(\begin{array}{ll} u = \arcsin t & du = \frac{1}{\sqrt{1-t^2}}dt \\ dv = t\,dt & v = t^2\frac{1}{2}\end{array}\right) = \frac{1}{2}t^2\arcsin t - \frac{1}{2}\int\frac{t^2dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{2}t^2\arcsin t - \frac{1}{2}\cdot\int\frac{t^2-1+1}{\sqrt{1-t^2}}dt = \frac{1}{2}t^2\arcsin t - \frac{1}{2}\left(-\int\sqrt{1-t^2}dt + \int\frac{dt}{\sqrt{1-t^2}}\right)$$

$$= \frac{1}{2}t^2\arcsin t + \frac{1}{2}\int\sqrt{1-t^2}dt - \frac{1}{2}\arcsin t = \frac{1}{2}t^2\arcsin t - \frac{1}{2}\arcsin t + \frac{1}{4}\arcsin t + \frac{1}{4}t\sqrt{1-t^2} + C$$

$$\int\sqrt{1-t^2}dt = \left(\begin{array}{l} t = \sin m \\ dt = \cos m\,dm \end{array}\right)_{m\in(-\frac{\pi}{2},\frac{\pi}{2})} = \int\underbrace{|\cos m|}_{\cos m}\cdot\cos m\,dm = \int\frac{1+\cos 2m}{2}dm = \frac{1}{2}m + \frac{1}{4}\sin 2m + C$$

$$= \frac{1}{2}\cdot\arcsin t + \frac{1}{4}2\cdot t\cdot\sqrt{1-t^2} + C$$