* Интеграл облика $\int_a^b f(x)\log^k\frac{b-x}{x-a}\,dx$

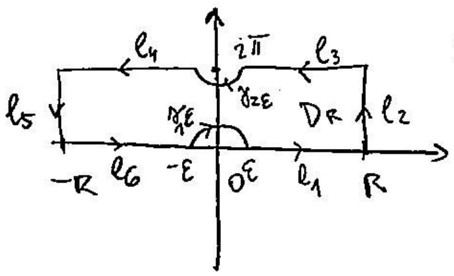
(где $f$ нема полове на $[a,b]$)

може да се израчуна слично као претходни с тим што се узима ф-ја $f(z)\log^{k+1}\frac{z-b}{z-a}$ и исти $\partial D$ као у претходној причи.

(3) Израчунати: $V.p.\int_{-\infty}^{+\infty}\frac{e^{ax}}{e^{2x}-1}\,dx,\ 0<a<2$

$$V.p.\int_{-\infty}^{+\infty}\frac{e^{ax}}{e^{2x}-1}\,dx=\lim_{R\to+\infty}\int_{-R}^{R}\frac{e^{ax}}{e^{2x}-1}\,dx$$

Нека је $f(z)=\frac{e^{az}}{e^{2z}-1}$ и $D_R$ област на слици




$\partial D_R=\Gamma=l_1+l_2+l_3+\gamma_{2\varepsilon}^-+l_4+l_5+l_6+\gamma_{1\varepsilon}^-$

$i\pi$ и $0$ морамо да заобиђемо јер крива не сме да прође кроз сингуларитет

$l_1(t)=t,\ t\in[\varepsilon,R]$

$l_2(t)=R+it,\ t\in[0,\pi]$

$l_3(t)=-t+i\pi,\ t\in[-R,-\varepsilon]$

$l_4(t)=-t+i\pi,\ t\in[\varepsilon,R]$

$l_5(t)=-R-it,\ t\in[-\pi,0]$

$l_6(t)=t,\ t\in[-R,-\varepsilon]$

$e^{2z}=1$ за $2z=2k\pi i$

$z=k\pi i$

Унутар $\mathrm{Int}\,\Gamma$ нема више сингуларитета

$\Rightarrow \int_\Gamma f(z)\,dz=0$

$$\lim_{z\to0} z f(z)=\lim_{z\to0}\frac{z\,e^{az}}{e^{2z}-1}=\frac{1}{2}\quad\text{јер}\quad\lim_{z\to0}\frac{e^{2z}-1}{2z}=1$$

$e^{2z}=1+2z+\frac{(2z)^2}{2!}+\dots$

$$\underset{\text{НЛЛ}}{\Longrightarrow}\ \lim_{\varepsilon\to0^+}\int_{\gamma_{1\varepsilon}}f(z)\,dz=-i\cdot\pi\cdot\frac{1}{2}=\frac{-i\pi}{2}$$

$$\lim_{z\to i\pi}(z-i\pi)f(z)=\lim_{z\to i\pi}\frac{e^{az}(z-i\pi)}{e^{2z}-1}=\lim_{z\to i\pi}\frac{e^{a(z-i\pi)}(z-i\pi)\cdot e^{ai\pi}}{e^{2(z-i\pi)}\cdot e^{2i\pi}-1}$$

$$\overset{w=z-i\pi}{=} \lim_{w\to 0} \frac{e^{aw}\cdot w\, e^{ai\pi}}{e^{2w}-1} = \frac{1}{2}\cdot e^{ai\pi}$$

$$\underset{ΜΛΛ}{\Rightarrow} \lim_{\varepsilon\to 0^+} \int_{\gamma_{2\varepsilon}} f(z)\,dz = -i\pi\cdot\frac{1}{2}e^{ai\pi} = \frac{-i\pi e^{ai\pi}}{2}$$

$$I = \int_{l_1} f(z)dz + \int_{l_2} f(z)dz + \int_{l_3} f(z)dz + \int_{l_4} f(z)dz + \int_{l_5} f(z)dz + \int_{l_6} f(z)dz$$
$$+ \int_{\gamma_{1\varepsilon}} f(z)dz + \int_{\gamma_{2\varepsilon}} f(z)dz$$

$$\int_{l_1} f(z)dz = \int_{\varepsilon}^{R} \frac{e^{at}}{e^{2t}-1}\,dt$$

$$\int_{l_2} f(z)dz = \int_0^{\pi} \frac{e^{a(R+it)}}{e^{2(R+it)}-1}\, i\,dt$$

$$\int_{l_4} f(z)dz = \int_{\varepsilon}^{R} \frac{e^{-at+ai\pi}}{e^{-2t+2i\pi}-1}(-dt)$$

$$= e^{ai\pi}\int_{\varepsilon}^{R} \frac{e^{-at}}{e^{-2t}-1}(-dt)$$

$u=-t$

$$= e^{ai\pi}\int_{-\varepsilon}^{-R} \frac{e^{au}}{e^{2u}-1}\,du = -e^{ai\pi}\int_{-R}^{-\varepsilon}\frac{e^{au}}{e^{2u}-1}\,du$$

$$\int_{l_3} f(z)dz = \int_{-R}^{-\varepsilon} \frac{e^{-at+ai\pi}}{e^{-2t+2i\pi}-1}(-dt)$$

$$= e^{ai\pi}\int_{-R}^{-\varepsilon}\frac{e^{-at}}{e^{-2t}-1}(-dt)$$

$u=-t,\ du=-dt$

$$= e^{ai\pi}\int_{R}^{\varepsilon}\frac{e^{au}}{e^{2u}-1}\,du$$

$$= e^{ai\pi}(-1)\int_{\varepsilon}^{R}\frac{e^{at}}{e^{2t}-1}\,dt$$

$$\int_{l_5} f(z)dz = \int_{-\pi}^{0} \frac{e^{a(-R-it)}}{e^{2(-R-it)}-1}(-i)\,dt$$

$$\int_{l_6} f(z)dz = \int_{-R}^{-\varepsilon}\frac{e^{at}}{e^{2t}-1}\,dt$$

$$\lim_{R\to+\infty}\int_0^{\pi}\frac{e^{aR}e^{ait}}{e^{2R}e^{2it}-1}\,dt = 0$$

jep:
$$\left|\frac{e^{aR}e^{ait}}{e^{2R}e^{2it}-1}\right| = \frac{e^{aR}}{|e^{2R}e^{2it}-1|} \le \underset{R\to+\infty}{\frac{e^{aR}}{e^{2R}-1}\to 0}$$ jep $a\in(0$

jep $|e^{2R}e^{2it}-1| \geq |e^{2R}e^{2it}|-1$

$$0 \le \left|\int_0^{\pi}\frac{e^{aR}e^{ait}}{e^{2R}e^{2it}-1}\,dt\right| \overset{\text{ολ}''}{\le} \int_0^{\pi}\frac{e^{aR}}{e^{2R}-1}\,dt \xrightarrow[R\to+\infty]{} 0$$

Слично је и $\lim\limits_{R\to\infty} \int_{-\pi}^{0} \frac{e^{a(-R)} e^{-ait}}{e^{-2R} e^{-2it} - 1} dt = 0.$

$$0 \le \left| \int_{-\pi}^{0} \frac{e^{-aR} e^{-ait}}{e^{-2R} e^{-2it} - 1} dt \right| \underset{\text{ОНН}}{\le} \int_{-\pi}^{0} \frac{e^{-aR}}{|e^{-2R} e^{-2it} - 1|} dt$$

$$\le \int_{-\pi}^{0} \frac{e^{-aR}}{e^{-2R} - 1} dt = \pi \cdot \frac{e^{-aR}}{e^{-2R} - 1} \xrightarrow[R\to\infty]{} 0$$

$$\underset{\substack{R\to+\infty \\ \varepsilon\to 0+}}{\lim} \Longrightarrow \quad 0 = \lim_{R\to+\infty} \left( \int_{0}^{R} \frac{e^{at}}{e^{2t} - 1} dt \cdot (1 - e^{ai\pi}) + \int_{-R}^{0} \frac{e^{at}}{e^{2t} - 1} dt \, (1 - e^{ai\pi}) \right) - \frac{i\pi}{2} (1 + e^{ai\pi})$$

$$i \frac{\pi}{2} (1 + e^{ai\pi}) = (1 - e^{ai\pi}) \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{at}}{e^{2t} - 1} dt$$

$$\text{v.p.} \int_{-\infty}^{+\infty} \frac{e^{at}}{e^{2t} - 1} dt = i \frac{\pi}{2} \; \frac{1 + e^{ai\pi}}{1 - e^{ai\pi}}$$

$$\begin{aligned} 1 + e^{ai\pi} &= 1 + \cos a\pi + i \sin a\pi \\ &= 2\cos^2 \frac{a\pi}{2} + 2i \sin \frac{a\pi}{2} \cos \frac{a\pi}{2} \\ &= 2\cos \frac{a\pi}{2} \cdot e^{i\frac{a\pi}{2}} \end{aligned}$$

$$\begin{aligned} 1 - e^{ai\pi} &= 1 - \cos a\pi - i \sin a\pi \\ &= 2\sin^2 \frac{a\pi}{2} - 2i \sin \frac{a\pi}{2} \cos \frac{a\pi}{2} \\ &= 2\sin \frac{a\pi}{2} \left( \sin \frac{a\pi}{2} - i \cos \frac{a\pi}{2} \right) \\ &= 2\sin \frac{a\pi}{2} \cdot (-i) \left( \cos \frac{a\pi}{2} + i \sin \frac{a\pi}{2} \right) \\ &= -2i \sin \frac{a\pi}{2} \, e^{i\frac{a\pi}{2}} \end{aligned}$$

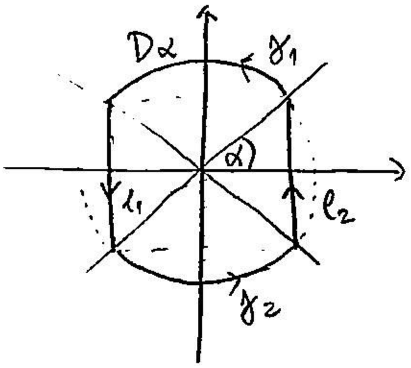
$$\Longrightarrow \frac{1 + e^{ai\pi}}{1 - e^{ai\pi}} = \frac{\cos \frac{a\pi}{2}}{\sin \frac{a\pi}{2}} \cdot i = \operatorname{ctg} \frac{a\pi}{2} \cdot i$$

$$\text{v.p} \int_{-\infty}^{+\infty} \frac{e^{at}}{e^{2t}-1}\,dt = i\frac{\pi}{2}\cdot i \operatorname{ctg}\frac{a\pi}{2} = \frac{-\pi}{2}\operatorname{ctg}\frac{a\pi}{2}$$

(4) Нека је $D_\alpha = \{ z \in \mathbb{C} : |z|<1 \,,\, |\mathrm{Re}\, z| < \cos\alpha \}$ , $0<\alpha<\frac{\pi}{2}$ .

Израчунати :
$$I(\alpha) = \int_{\partial D_\alpha} \frac{1}{iz} \log\frac{1-z^2}{2}\,dz$$

а затим наћи
$$I = \int_0^{\pi} \log(\sin t)\,dt .$$




$-\cos\alpha < \mathrm{Re}\, z < \cos\alpha$

$\partial D_\alpha = \gamma_1 + l_1 + \gamma_2 + l_2$

$\gamma_1(t) = e^{it} \,,\, t\in[\alpha, \pi-\alpha]$

$\gamma_2(t) = e^{it} \,,\, t\in[\alpha-\pi, -\alpha]$

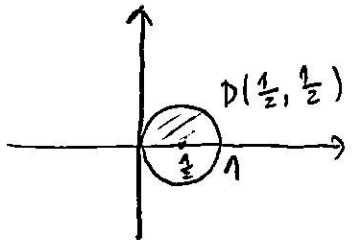
$l_1(t) = -\cos\alpha - it ,\, t\in[-\sin\alpha, \sin\alpha]$

$l_2(t) = \cos\alpha + it \,,\, t\in[-\sin\alpha, \sin\alpha]$

функција $\frac{1-z^2}{2}$ слика $\mathbb{D} = \{z\in\mathbb{C} : |z|<1\}$ у $D(\frac{1}{2},\frac{1}{2})$

јер $z\in\mathbb{D} \Rightarrow z^2\in\mathbb{D} \Rightarrow -z^2\in\mathbb{D} \Rightarrow 1-z^2 \in D(1,1)$

$D(1,1) = \{ w\in\mathbb{C} : |w-1|<1 \}$




$\Rightarrow \frac{1-z^2}{2} \in D(\frac{1}{2},\frac{1}{2})$

јер $\left|\frac{1-z^2}{2} - \frac{1}{2}\right| = \left|\frac{z^2}{2}\right| < \frac{1}{2}$

$$\log\frac{1-z^2}{2} = \log\left|\frac{1-z^2}{2}\right| + i\cdot\arg\frac{1-z^2}{2}$$

$\arg\frac{1-z^2}{2} \in (-\pi, \pi)$

овако бирамо грану!

функција $f(z) = \frac{1}{iz}\log\frac{1-z^2}{2}$

има пол у 0 реда 1 у $D_\alpha$

$$\Rightarrow \int_{\partial D_\alpha} f(z)\,dz = 2\pi i\cdot \mathrm{Res}(f,0) = I(\alpha)$$

$$\operatorname{Res}(f,0) = \lim_{z\to 0} z \cdot \frac{1}{iz} \log \frac{1-z^2}{2} = \frac{1}{i} \log \frac{1}{2} = i \log 2$$

$$I(\alpha) = 2\pi i \cdot (i \log 2) = -2\pi \log 2$$

Са друге стране,

$$I(\alpha) = \int_{\gamma_1} f(z)\,dz + \int_{l_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{l_2} f(z)\,dz$$

$$\int_{\gamma_1} f(z)\,dz = \int_{\alpha}^{\pi-\alpha} \frac{1}{ie^{it}} \log \frac{1-e^{2it}}{2}\, ie^{it}\,dt$$

$$= \int_{\alpha}^{\pi-\alpha} \log \frac{1-e^{2it}}{2}\,dt$$

$$\int_{\gamma_2} f(z)\,dz = \int_{\alpha-\pi}^{-\alpha} \frac{1}{\cancel{ie^{it}}} \log \frac{1-e^{2it}}{2}\, \cancel{ie^{it}}\,dt$$

$$= \int_{\alpha-\pi}^{-\alpha} \log \frac{1-e^{2it}}{2}\,dt$$

$$+\left\{ \begin{aligned} \int_{l_1} f(z)\,dz &= \int_{-\sin\alpha}^{\sin\alpha} \frac{1}{i(-\cos\alpha - it)} \cdot \log \frac{1-(\cos\alpha+it)^2}{2} \cdot i\,dt \\ &= \int_{-\sin\alpha}^{\sin\alpha} \frac{-1}{\cos\alpha+it} \log \frac{\sin^2\alpha - 2it\cos\alpha + t^2}{2}\,dt \\ \int_{l_2} f(z)\,dz &= \int_{-\sin\alpha}^{\sin\alpha} \frac{1}{i(\cos\alpha+it)} \log \frac{1-(\cos\alpha+it)^2}{2}\, i\,dt \\ &= \int_{-\sin\alpha}^{\sin\alpha} \frac{1}{\cos\alpha+it} \log \frac{\sin^2\alpha - 2it\cos\alpha + t^2}{2}\,dt \end{aligned} \right.$$

$$\int_{l_1} f(z)\,dz + \int_{l_2} f(z)\,dz = 0 \quad \text{(скрате се)}$$

$$I(\alpha) = -2\pi\log 2 = \int_{\alpha}^{\pi-\alpha} \log\frac{1-e^{2it}}{2}\,dt + \int_{\alpha-\pi}^{-\alpha} \log\frac{1-e^{2it}}{2}\,dt$$

$$1-e^{2it} = -2i\sin t\; e^{it}$$

$$\frac{1-e^{2it}}{2} = -i\sin t\, e^{it}$$

$\arg\frac{1-e^{2it}}{2} \in (-\pi,\pi]$ !

За $t\in[\alpha,\pi-\alpha]$: $\log(-i\sin t\, e^{it}) = \log(\sin t) + \underset{\log|ie^{it}|+i\arg(-ie^{it})}{\log(-ie^{it})} = \log(\sin t) + i(t-\frac{\pi}{2})$

$$-ie^{it} = e^{-i\frac{\pi}{2}}e^{it} = e^{i(t-\frac{\pi}{2})}$$

за $t\in(0,\pi)$ $\quad t-\frac{\pi}{2}\in(-\frac{\pi}{2},\frac{\pi}{2})$

за $t\in[\alpha-\pi,-\alpha]$ је $\sin t<0$, па је

$$\log\frac{1-e^{2it}}{2} = \log|\sin t| + \underset{\log|ie^{it}|+i\arg(ie^{it})}{\log(ie^{it})} = \log(-\sin t) + (t+\frac{\pi}{2})i$$

$$ie^{it} = e^{i\frac{\pi}{2}}e^{it} = e^{i(t+\frac{\pi}{2})}$$

за $t\in(-\pi,0)$ $\quad t+\frac{\pi}{2}\in(-\frac{\pi}{2},\frac{\pi}{2})$

$$-2\pi\log 2 = \int_{\alpha}^{\pi-\alpha}\left(\log(\sin t) + i(t-\tfrac{\pi}{2})\right)dt + \int_{\alpha-\pi}^{-\alpha}\left(\log(-\sin t) + i(t+\tfrac{\pi}{2})\right)dt$$

$$= \int_{\alpha}^{\pi-\alpha}\log(\sin t)\,dt + i\left(\frac{t^2}{2}-\frac{\pi}{2}t\right)\Big|_{\alpha}^{\pi-\alpha} + \underbrace{\int_{\alpha-\pi}^{-\alpha}\log(-\sin t)\,dt}_{\text{смена } u=-t,\ du=-dt}$$

$$+ i\left(\frac{t^2}{2}+\frac{\pi}{2}t\right)\Big|_{t=\alpha-\pi}^{-\alpha}$$

$$= \int_{\alpha}^{\pi-\alpha}\log(\sin t)\,dt + i\left(\cancel{\frac{(\pi-\alpha)^2}{2}} - \frac{\pi}{2}(\cancel{\pi}-\alpha) - \cancel{\frac{\alpha^2}{2}} + \cancel{\frac{\pi}{2}\alpha}\right)$$

$$+ \int_{\pi-\alpha}^{\alpha}\log(\sin u)(-du) + i\left(\frac{\alpha^2}{2} + \frac{\pi}{2}(-\alpha) - \cancel{\frac{(\alpha-\pi)^2}{2}} - \cancel{\frac{\pi}{2}}(\alpha-\pi)\right)$$

$$= 2\cdot\int_{\alpha}^{\pi-\alpha}\log(\sin t)\,dt \qquad /\lim_{\alpha\to 0^+}$$

$$\Rightarrow \boxed{\int_0^{\pi}\log(\sin t)\,dt = -\pi\log 2}$$