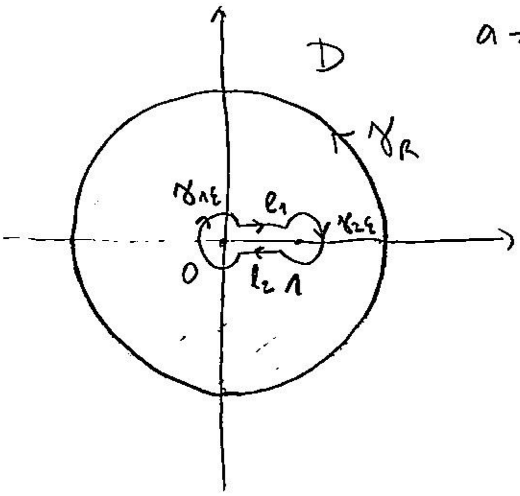
① Израчунати: $I=\int_0^1 \frac{x^{\alpha}(1-x)^{1-\alpha}}{1+x^2}\,dx$, $1<\alpha<2$

$0<\alpha-1<1$

$$= \int_0^1 \frac{x\, x^{\alpha-1}}{(1+x^2)\,(1-x)^{\alpha-1}}\,dx$$

$$= \int_0^1 \frac{x}{1+x^2}\left(\frac{x}{1-x}\right)^{\alpha-1}dx$$



$a=0,\ b=1$, $f(z)=\frac{z}{z^2+1}$

$$\rho(z)=\frac{z}{z-1}$$

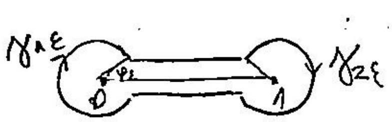
$$(\rho(z))^{\alpha-1}=\left|\frac{z}{z-1}\right|^{\alpha-1} e^{i\arg\frac{z}{z-1}(\alpha-1)}$$

$$\arg\frac{z}{z-1}\in(-\pi,\pi)$$

ρ дефинисана на $\mathbb{C}\setminus[0,1]$

$$g(z)=\frac{z}{z^2+1}\cdot\left|\frac{z}{z-1}\right|^{\alpha-1} e^{i\arg\frac{z}{z-1}(\alpha-1)}$$

$$\Gamma=\partial D=\gamma_R\cup(\gamma_{1\varepsilon}+l_1+\gamma_{2\varepsilon}+l_2)$$



$$\sin\varphi_\varepsilon = h/\varepsilon$$

$$\varphi_\varepsilon=\arcsin\frac{h}{\varepsilon}$$

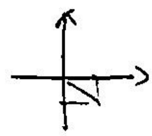
$$J=\int_\Gamma g(z)\,dz = 2\pi i\,(\mathrm{Res}(g,i)+\mathrm{Res}(g,-i))$$

$$\mathrm{Res}(g,i)=\lim_{z\to i}\frac{z}{z+i}\left|\frac{z}{z-1}\right|^{\alpha-1} e^{i\arg\frac{z}{z-1}(\alpha-1)}$$

$$=\frac{i}{2i}\cdot\left|\frac{i}{i-1}\right|^{\alpha-1} e^{i\arg\frac{i}{i-1}(\alpha-1)}$$

$$\frac{i}{i-1}\cdot\frac{-1-i}{-1-i}=\frac{-i+1}{2}$$

$$\arg\frac{i}{i-1}=-\frac{\pi}{4}$$



$$\left|\frac{i}{i-1}\right|=\frac{1}{\sqrt2}$$

$$\mathrm{Res}(g,i) = \frac{1}{2}\cdot\left(\frac{1}{\sqrt{2}}\right)^{\alpha-1}\cdot e^{i\frac{-\pi}{4}(\alpha-1)} = \frac{1}{2^{\frac{\alpha-1}{2}+1}}\, e^{-i\frac{\pi}{4}(\alpha-1)}$$

$$\mathrm{Res}(g,-i) = \lim_{z\to -i} \frac{z}{z-i}\left|\frac{z}{z-1}\right|^{\alpha-1}\cdot e^{i\arg\frac{z}{z-1}\cdot(\alpha-1)}$$

$$= \frac{-i}{-2i}\cdot\left|\frac{-i}{-i-1}\right|^{\alpha-1}\cdot e^{i\arg\frac{-i}{-i-1}(\alpha-1)}$$

$$\frac{-i}{-i-1} = \frac{i}{i+1}\cdot\frac{1-i}{1-i} = \frac{i+1}{2}$$

$$\arg\frac{i+1}{2} = \frac{\pi}{4}$$

$$\left|\frac{i+1}{2}\right| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\mathrm{Res}(g,-i) = \frac{1}{2}\cdot\frac{1}{(\sqrt{2})^{\alpha-1}}\cdot e^{i\frac{\pi}{4}(\alpha-1)}$$

$$J = 2\pi i\cdot\left(\frac{1}{2^{\frac{\alpha-1}{2}+1}}\cdot\left(e^{i\frac{\pi}{4}(\alpha-1)} + e^{-i\frac{\pi}{4}(\alpha-1)}\right)\right)$$

$$J = 2\pi i\cdot\frac{1}{2^{\frac{\alpha+1}{2}}}\, 2\cos\frac{\pi}{4}(\alpha-1)$$

$$\boxed{J = 2\pi i\cdot\frac{1}{2^{\frac{\alpha-1}{2}}}\cos\frac{\pi(\alpha-1)}{4}}$$

$$\lim_{z\to 0} z g(z) = 0 \underset{\text{ЈЛЛ 1}}{\Longrightarrow} \lim_{\varepsilon\to 0^+}\int_{\gamma_\varepsilon} g(z)\,dz = 0$$

$$\lim_{z\to 1} (z-1) g(z) = 0 \underset{\text{ЈЛЛ 1.}}{\Longrightarrow} \lim_{\varepsilon\to 0^+}\int_{\gamma_{2\varepsilon}} g(z)\,dz = 0$$

$$l_1(t) = t + ih,\quad t\in\left[\sqrt{\varepsilon^2-h^2},\, 1-\sqrt{\varepsilon^2-h^2}\right]$$

$$l_2(t) = -t - ih,\quad t\in\left[-1+\sqrt{\varepsilon^2-h^2},\, -\sqrt{\varepsilon^2-h^2}\right]$$

као у претх. задатку:

$$\text{на } l_1 \quad \arg\frac{z}{z-1} \xrightarrow[h\to 0^+]{} -\pi$$

$$\text{на } l_2 \quad \arg\frac{z}{z-1} \xrightarrow[h\to 0^+]{} \pi$$

$$\Rightarrow \lim_{h\to 0^+}\int_{l_1} g(z)\,dz = \int_{\varepsilon}^{1-\varepsilon} f(t)\left(\frac{t}{1-t}\right)^{\alpha-1} e^{i(\alpha-1)\pi}\,dt$$

$$\lim_{h\to 0^+}\int_{l_2} g(z)\,dz = \int_{-1+\varepsilon}^{-\varepsilon} -f(-t)\left(\frac{-t}{1+t}\right)^{\alpha-1} e^{i(\alpha-1)\pi}\,dt$$

$$\xrightarrow[\varepsilon\to 0^+]{} -\int_0^1 f(t)\left(\frac{t}{1-t}\right)^{\alpha-1} dt\cdot e^{i(\alpha-1)\pi}$$

$$\int_{l_1} g(z)\,dz + \int_{l_2} g(z)\,dz \xrightarrow[\varepsilon\to 0+]{h\to 0+} \underbrace{\int_0^1 f(t)\left(\frac{t}{1-t}\right)^{\alpha-1} dt \cdot \left(e^{-i(\alpha-1)\pi} - e^{i(\alpha-1)\pi}\right)}_{\substack{\| \\ -2i\sin(\alpha-1)\pi \cdot I}}$$

$$\int_{\gamma_R} g(z)\,dz = -2\pi i \cdot \operatorname{Res}(g, \infty)$$

$\operatorname{Res}(g,\infty) = -a_{-1}$ $\quad a_{-1}$ коеф уз $\frac{1}{z}$ у Лорановом развоју

$$g(z) = \frac{z}{z^2+1} \cdot \left(\frac{z}{z-1}\right)^{\alpha-1}$$

$$z = \frac{1}{u}$$

$$g\left(\frac{1}{u}\right) = \frac{1}{u} \cdot \frac{1}{\frac{1}{u^2}+1} \cdot \left(\frac{\frac{1}{u}}{\frac{1}{u}-1}\right)^{\alpha-1}$$

$$= \frac{u}{u^2+1} \cdot \left(\frac{1}{1-u}\right)^{\alpha-1}$$

$$= u \cdot \underbrace{\sum_{n=0}^{\infty} (-1)^n u^{2n}}_{\substack{\| \\ (1-u^2+\dots)}} \cdot \underbrace{\sum_{n=0}^{\infty} (-1)^n \binom{1-\alpha}{n} u^n}_{\substack{\| \\ (1-(1-\alpha)u+\dots)}}$$

уз u je 1

$$a_{-1} = 1$$

$$\Rightarrow \quad \int_{\gamma_R} g(z)\,dz = -2\pi i \cdot (-1) = 2\pi i$$

$$\Rightarrow \quad \cancel{2}\pi \cancel{i} \cdot 2^{\frac{1-\alpha}{2}} \cos\frac{\pi(\alpha-1)}{4} = \cancel{2}\pi \cancel{i} - I \cdot \cancel{2i}\sin((\alpha-1)\pi)$$

$$\pi\, 2^{\frac{1-\alpha}{2}} \cos\frac{\pi(\alpha-1)}{4} = \pi - I \cdot \sin(\alpha-1)\pi$$

$$\sin(\alpha-1)\pi = \sin\alpha\pi\,(-1)$$

$$\pi\left(2^{\frac{1-\alpha}{2}}\cos\frac{\pi(\alpha-1)}{4}-1\right)=I\sin\alpha\pi$$

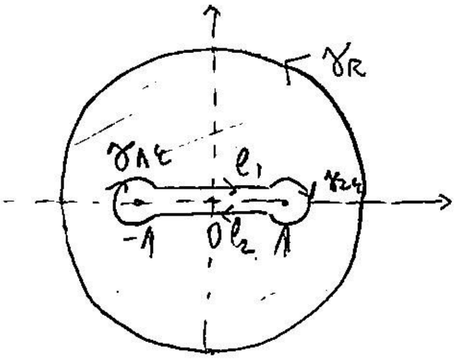
$$\Rightarrow \boxed{I=\frac{\pi}{\sin\alpha\pi}\cdot\left(2^{\frac{1-\alpha}{2}}\cos\frac{\pi(\alpha-1)}{4}-1\right)}$$

(2) Израчунати: $I=\int_{-1}^{1}\frac{\sqrt[5]{(1+x)(1-x)^4}}{1+x^2}\,dx$

$$=\int_{-1}^{1}\frac{1-x}{1+x^2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{5}}dx$$

$f(z)=\frac{1-z}{1+z^2}$ (полови $\pm i$ реда 1)

$s(z)=\frac{z+1}{z-1}$



$\varphi_\varepsilon=\arcsin\frac{h}{\varepsilon}$

$$g(z)=\left|\frac{z+1}{z-1}\right|^{\frac{1}{5}}\cdot e^{i\arg\frac{z+1}{z-1}\cdot\frac{1}{5}}\cdot\frac{1-z}{1+z^2}$$

$\arg\frac{z+1}{z-1}\in(-\pi,\pi)$

g деф на $\mathbb{C}\setminus[-1,1]$

$l_1(t)=t+ih,\ t\in\left[-1+\sqrt{\varepsilon^2-h^2},\ 1-\sqrt{\varepsilon^2-h^2}\right]$

$l_2(t)=-t-ih,\ t\in\left[-1+\sqrt{\varepsilon^2-h^2},\ 1-\sqrt{\varepsilon^2-h^2}\right]$

$\gamma_{1\varepsilon}(t)=-1+\rho e^{it},\ t\in[\varphi_\varepsilon,\ 2\pi-\varphi_\varepsilon]$

$\gamma_{2\varepsilon}(t)=1+\rho e^{it},\ t\in[\pi+\varphi_\varepsilon,\ 3\pi-\varphi_\varepsilon]$

Нпр. али ово нам није битно јер користимо Јорданове леме!

$\Gamma=\gamma_R\cup(\gamma_{1\varepsilon}+l_1+\gamma_{2\varepsilon}+l_2)$

$$\int_\Gamma g(z)\,dz=2\pi i\cdot\left(\operatorname{Res}(g,i)+\operatorname{Res}(g,-i)\right)$$

итд све аналогно!

резултат:

$$I=\frac{\pi}{\sin\frac{\pi}{5}}\left(-1+\cos\frac{\pi}{10}+\sin\frac{\pi}{10}\right)$$