* Интеграл облика $\int_0^{+\infty} R(x)(\log x)^m dx$

$f(z) = R(z)(\log z)^{m+1}$ по „C" контури ако

R нема полова на $\mathbb{R}^+$ (ако је R и парна може и)

* Интеграл облика $J(\alpha, p) = \int_0^{+\infty} x^{\alpha} R(x) (\ln x)^p dx, \ p \in \mathbb{N}, \ -1 < \alpha < 0$

(3) Израчунати $\int_0^{+\infty} \frac{1}{\sqrt{x}(1+x^2)} \log x \, dx$ (домаћи)

(рађен сличан)

$$f(z) = \frac{\log z}{g(z)(1+z^2)}$$

$$\log z = \log|z| + i \arg z \quad , \arg z \in (0, 2\pi)$$

$$g(z) = \sqrt{|z|} \cdot e^{i \frac{\arg z}{2}} \quad , \arg z \in (0, 2\pi)$$

по „C" контури резултат: $-\frac{\sqrt{2}}{4}\pi^2$

* Интеграл облика $\int_a^b f(x)\left(\frac{x-a}{b-x}\right)^{\alpha} dx \quad , 0 < \alpha < 1$

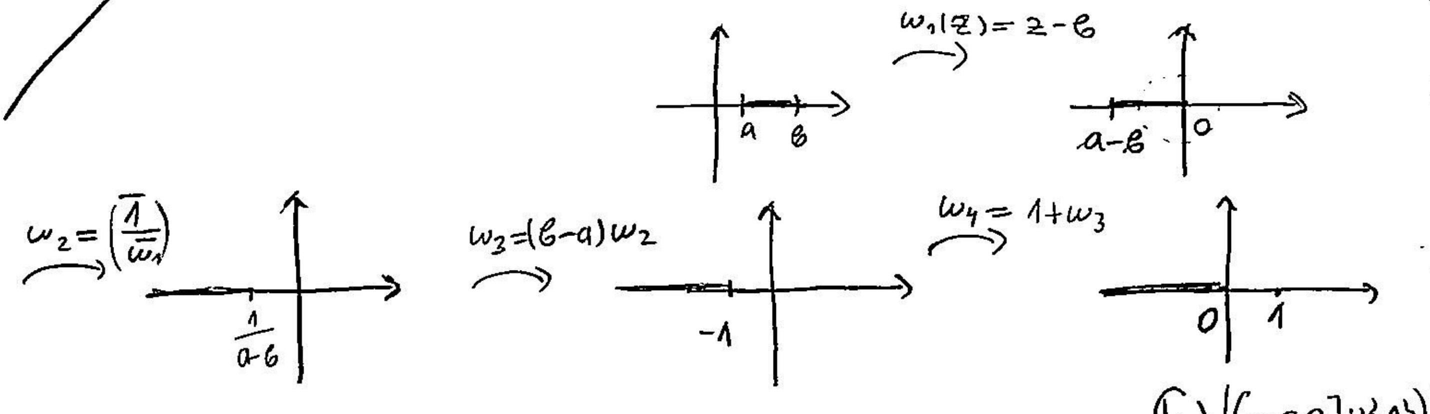
$$f \in H(\mathbb{C} \setminus \{z_1, \dots, z_n\})$$

$z_1, \dots, z_n \notin [a, b]$ и f у a има највише пол 1. реда

$\rho = \frac{z-a}{z-b}$ слика $\mathbb{C} \setminus [a, b]$

на $\mathbb{C} \setminus ((-\infty, 0] \cup \{1\})$

доказ: $\rho(z) = \frac{z-a}{z-b} = \frac{z-b+b-a}{z-b} = 1 + \frac{b-a}{z-b}$



$\rho(z) = \omega$

$z - a = \omega(z - b)$

$z(1 - \omega) = a - \omega b$

$$z = \frac{\omega b - a}{\omega - 1}$$

$$\rho^{-1}(\omega) = \frac{\omega b - a}{\omega - 1}$$

ρ је бијекција $\overline{\mathbb{C}}$ $\quad \rho(b)=\infty$, $\rho(\infty)=1$

ρ слика $\mathbb{C}\setminus\{b\}$ бијективно на $\mathbb{C}\setminus\{1\}$

зато $\mathbb{C}\setminus[a,b]$ слика на $\mathbb{C}\setminus\left((-\infty,0]\cup\{1\}\right)$

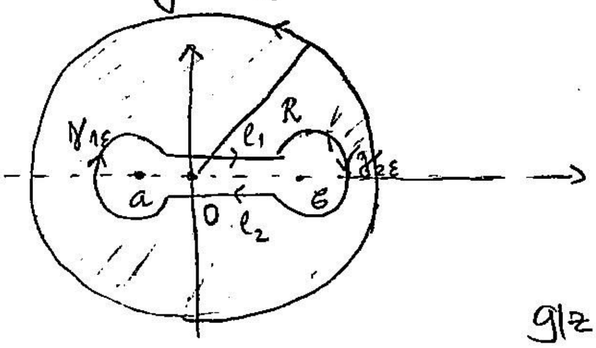
(ρ слика $[a,b)$ на $(-\infty,0]$)

$$z^{\alpha} = e^{\alpha \log z} \qquad \log z = \log|z| + i\arg z, \quad \arg z \in (-\pi,\pi)$$

$$z^{\alpha} = e^{\alpha\log|z| + \alpha i \arg z} = |z|^{\alpha}\cdot e^{i\arg z\cdot\alpha}$$

$$(\rho(z))^{\alpha} = \left(\frac{z-a}{z-b}\right)^{\alpha}$$ Пошто $\rho:\mathbb{C}\setminus[a,b]\to\mathbb{C}\setminus\left((-\infty,0]\cup\{1\}\right)$

узимамо грану ф-је $\rho^{\alpha} = |\rho|^{\alpha} e^{i\arg\rho\,\alpha}$, т.д. $\arg\rho\in(-\pi,\pi)$



D осенчена област

$$\partial D = \gamma_R \cup (\gamma_1^- + l_1 + \gamma_2^- + l_2)$$

интегралимо ф-ју

$$g(z) = f(z)\cdot|\rho(z)|^{\alpha} e^{i\arg\rho(z)\cdot\alpha}$$

где $\arg\rho(z)\in(-\pi,\pi)$

по ∂D!

$$\int_{\partial D} g(z)\,dz = 2\pi i\sum_{k=1}^{n}\mathrm{Res}(g,z_k)$$

$z_k \neq a$ за све k!

Даље ћемо размотрити на конкретном примеру.

(4) Израчунати: $I = \int_0^1 \frac{\sqrt[3]{x^5(1-x)}}{1+x^2}\,dx = \int_0^1 \frac{x\sqrt[3]{x^2(1-x)}}{1+x^2}\,dx$

$$I = \int_0^1 \frac{x\cdot\sqrt[3]{\frac{x^2}{(1-x)^2}}\,(1-x)}{1+x^2}\,dx$$

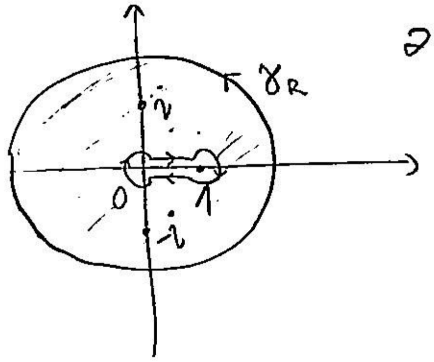
$$I = \int_0^1 \frac{x(1-x)}{1+x^2}\left(\frac{x}{1-x}\right)^{\frac{2}{3}}dx$$

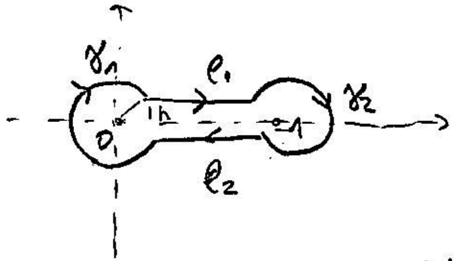
$$f(z) = \frac{z(1-z)}{1+z^2}$$

$a=0$, $b=1$, $\alpha=\frac{2}{3}$

$$g(z) = \frac{z(1-z)}{1+z^2}\cdot\left|\frac{z}{z-1}\right|^{\frac{2}{3}} e^{i\arg\frac{z}{z-1}\cdot\frac{2}{3}}, \quad \arg\frac{z}{z-1}\in(-\pi,\pi)$$

$$\partial D = \gamma_R \cup (\gamma_{1\varepsilon} + l_1 + \gamma_{2\varepsilon} + l_2)$$





$$\arcsin\frac{h}{\varepsilon} = \varphi_\varepsilon$$

(Кад обилазимо границу области остаје са леве стране)

$$\gamma_{1\varepsilon}(t) = \varepsilon e^{it}, \quad t \in [\varphi_\varepsilon, 2\pi - \varphi_\varepsilon]$$

$$\gamma_{2\varepsilon}(t) = 1 + \varepsilon e^{it}, \quad t \in [\pi + \varphi_\varepsilon, 3\pi - \varphi_\varepsilon]$$

$$l_1(t) = t + ih, \quad t \in [\sqrt{\varepsilon^2 - h^2}, 1 - \sqrt{\varepsilon^2 - h^2}]$$

$$l_2(t) = -t - ih, \quad t \in [-1 + \sqrt{\varepsilon^2 - h^2}, -\sqrt{\varepsilon^2 - h^2}]$$

$$\int_{\partial D} g(z)\,dz = 2\pi i \cdot (\mathrm{Res}(g, i) + \mathrm{Res}(g, -i))$$

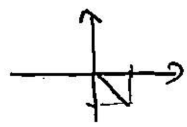
$\pm i$ полови првог реда за g

$$\mathrm{Res}(g, i) = \lim_{z \to i} \frac{z(1-z)}{z+i} \left|\frac{z}{z-1}\right|^{\frac{2}{3}} e^{i \arg\frac{z}{z-1} \cdot \frac{2}{3}}$$

$$= \frac{i(1-i)}{2i} \left|\frac{i}{i-1}\right|^{\frac{2}{3}} e^{i \arg\frac{i}{i-1} \cdot \frac{2}{3}}$$

$$\frac{i}{i-1} \cdot \frac{-1-i}{-1-i} = \frac{-i+1}{2} = \frac{1}{2} - \frac{i}{2}$$

$$= \frac{1-i}{2} \cdot \left(\frac{\sqrt{2}}{2}\right)^{\frac{2}{3}} e^{i \frac{-\pi}{4} \frac{2}{3}}$$



$$= \frac{1-i}{2} \cdot \frac{1}{\sqrt[3]{2}} e^{-\frac{i\pi}{6}}$$

$$\arg\left(\frac{i}{i-1}\right) = \frac{-\pi}{4}$$

$$= \frac{1}{2\sqrt[3]{2}} \cdot (1-i)\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$\left|\frac{i}{i-1}\right| = \frac{\sqrt{2}}{2}$$

$$\mathrm{Res}(g, -i) = \lim_{z \to -i} \frac{z(1-z)}{z-i} \left|\frac{z}{z-1}\right|^{\frac{2}{3}} \cdot e^{i \arg\frac{z}{z-1} \cdot \frac{2}{3}}$$

$$\frac{-i}{-i-1} \cdot \frac{-1+i}{-1+i} = \frac{i+1}{2}$$

$$\left|\frac{i+1}{2}\right| = \frac{\sqrt{2}}{2}$$

$$\mathrm{Res}(g, -i) = \frac{-i(1+i)}{-2i} \left(\frac{\sqrt{2}}{2}\right)^{\frac{2}{3}} \cdot e^{i \frac{\pi}{4} \frac{2}{3}}$$

$$\arg\frac{i+1}{2} = \frac{\pi}{4}$$

$$\operatorname{Res}(g,-i) = \frac{1+i}{2}\cdot\sqrt[3]{\frac{1}{2}}\; e^{i\frac{\pi}{6}} = \frac{1}{2\sqrt[3]{2}}\cdot(1+i)\cdot\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$$

$$\operatorname{Res}(g,i)+\operatorname{Res}(g,-i) = \frac{1}{2\sqrt[3]{2}}\cdot\left[\frac{\sqrt{3}}{2} - \cancel{\frac{i\sqrt{3}}{2}} - \cancel{\frac{i}{2}} - \frac{1}{2} + \frac{\sqrt{3}}{2} + \cancel{\frac{i\sqrt{3}}{2}} + \cancel{\frac{i}{2}} - \frac{1}{2}\right]$$

$$\operatorname{Res}(g,i)+\operatorname{Res}(g,-i) = \frac{1}{2\sqrt[3]{2}}\cdot(\sqrt{3}-1)$$

$$\int_{\gamma_R} g(z)\,dz + \int_{\gamma_{1\varepsilon}^-} g(z)\,dz + \int_{l_1} g(z)\,dz + \int_{l_2} g(z)\,dz + \int_{\gamma_{2\varepsilon}^-} g(z)\,dz = I$$

$$\lim_{z\to 0} z\,g(z) = \lim_{z\to 0} z\cdot\frac{z(1-z)}{1+z^2}\left|\frac{z}{z-1}\right|^{\frac{2}{3}} e^{i\arg\frac{z}{z-1}\cdot\frac{2}{3}} = 0$$

$$\lim_{z\to 1}(z-1)g(z) = \lim_{z\to 1}(z-1)\frac{z(1-z)}{1+z^2}\left|\frac{z}{z-1}\right|^{\frac{2}{3}} e^{i\arg\frac{z}{z-1}\cdot\frac{2}{3}} = 0$$

$$\underset{\text{Л1}}{\Longrightarrow}\quad \lim_{\varepsilon\to 0^+}\int_{\gamma_{1\varepsilon}^-} g(z)\,dz = \lim_{\varepsilon\to 0^+}\int_{\gamma_{2\varepsilon}^-} g(z)\,dz = 0$$

$$\lim_{h\to 0^+}\int_{l_1} g(z)\,dz = \int_{\varepsilon}^{1-\varepsilon} f(t)\cdot\left(\frac{t}{1-t}\right)^{\frac{2}{3}}\cdot e^{-i\pi\cdot\frac{2}{3}}\,dt$$

јер на l_1: $\left|\frac{z}{z-1}\right| = \left|\frac{t+ih}{t+ih-1}\right| \xrightarrow[h\to 0^+]{} \frac{t}{1-t} \quad (t>0,\ t<1)$

$$\arg\frac{z}{z-1} = \arg\frac{t+ih}{t+ih-1} \longrightarrow \arg\frac{t}{t-1} = -\pi \qquad \frac{t}{t-1}\in\mathbb{R}^-$$

$$\lim_{h\to 0^+}\int_{l_2} g(z)\,dz = \int_{-1+\varepsilon}^{-\varepsilon} f(-t)\left(\frac{-t}{1+t}\right)^{\frac{2}{3}} e^{i\pi\frac{2}{3}}(-dt)$$

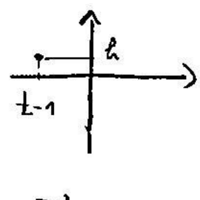
због $z=-t-ih$, $dz=-dt$

јер на l_2: $\left|\frac{z}{z-1}\right| = \left|\frac{-t-ih}{-t-ih-1}\right| \xrightarrow[h\to 0^+]{} \frac{-t}{1+t} \quad (t<0)$

$$\arg\frac{z}{z-1} = \arg\frac{-t-ih}{-t-ih-1} \xrightarrow[h\to 0^+]{} \pi \qquad (t<0,\ t>-1)$$

објашњење:

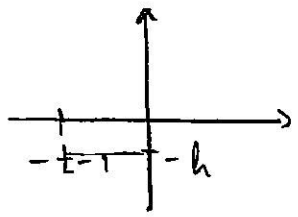
$$\frac{t+ih}{t+ih-1} = 1 + \frac{1}{t-1+ih}, \quad t-1<0,\ h>0$$



$$\arg(t-1+ih) \xrightarrow[h\to 0^+]{} \pi$$

$$\Rightarrow \arg\frac{1}{t-1+ih} \xrightarrow[h\to 0^+]{} -\pi$$

$$\frac{-t-ih}{-t-ih-1} = 1 + \frac{1}{-t-ih-1}$$



$$0 < -t < 1$$

$$-t-1 < 0$$

$$-h < 0 \quad \Rightarrow \arg(-t-ih-1) \xrightarrow{h \to 0^+} -\pi$$

$$\Rightarrow \arg \frac{1}{-t-ih-1} \xrightarrow[h\to 0^+]{} \pi$$

($\arg z = -\arg \frac{1}{z}$ јер

за $z = \rho e^{i\varphi}$ је

$\frac{1}{z} = \frac{1}{\rho} e^{-i\varphi}$)

на $(-\pi, \pi)$

Остаје још само да се израчуна $\int_{\gamma_R} g(z)dz$.

g је холоморфна на $\mathbb{C} \setminus B(0,R)$

$$\Rightarrow \int_{\gamma_R} g(z)dz = -2\pi i \cdot \mathrm{Res}(g,\infty)$$

$\mathrm{Res}(g,\infty)$ налазимо тако што развијемо g у Лоранов ред око ∞!

$$\mathrm{Res}(g,\infty) = -a_{-1}$$

$$g(z) = \frac{z(1-z)}{1+z^2}\left(\frac{z}{z-1}\right)^{\frac{2}{3}}$$

$z = \frac{1}{u}$, $|u|<1$ нпр.

$$g\left(\frac{1}{u}\right) = \frac{1}{u}\cdot\left(1-\frac{1}{u}\right)\frac{1}{1+\frac{1}{u^2}} \cdot \left(\frac{\frac{1}{u}}{\frac{1}{u}-1}\right)^{\frac{2}{3}}$$

$$g\left(\frac{1}{u}\right) = \frac{1}{u} \cdot \frac{u-1}{u} \cdot \frac{u^2}{1+u^2} \cdot \left(\frac{1}{1-u}\right)^{\frac{2}{3}}$$

$$= \frac{u-1}{1+u^2} \cdot \left(\frac{1}{1-u}\right)^{\frac{2}{3}}$$

$\binom{\alpha}{0} = 1$

$$(1-z)^{\alpha} = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} z^n$$ конв на $B(0,1)$ тј за $|z|<1$

$$(1-z)^{\alpha} = 1 - \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 + \dots$$

$$g\left(\frac{1}{u}\right) = -\frac{1-u}{1+u^2} \cdot (1-u)^{-\frac{2}{3}} = -\frac{1-u}{1+u^2} \cdot \left(1 + \frac{2}{3}u + \dots\right)$$

$$= -(1-u) \cdot \sum_{n=0}^{\infty} (-1)^n u^{2n} \left(1 + \frac{2}{3}u + \dots\right) = -(1-u)(1-u^2+\dots)(1+\frac{2}{3}$$

уз u је $1 - \frac{2}{3} = \frac{1}{3}$

$$\Rightarrow \operatorname{Res}(g, \infty) = -\frac{1}{3}$$

$$\Rightarrow \int_{\gamma_R} g(z)\,dz = -2\pi i \cdot \frac{-1}{3} = \frac{2\pi i}{3}$$

Коначно:

$$2\pi i \cdot \frac{\sqrt{3}-1}{2\sqrt[3]{2}} = \frac{2\pi i}{3} + \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1-\varepsilon} f(t) \left(\frac{t}{1-t}\right)^{\frac{2}{3}} dt \cdot e^{-\frac{2i\pi}{3}}$$

$$+ \lim_{\varepsilon \to 0^+} \int_{-1+\varepsilon}^{-\varepsilon} -f(-t) \left(\frac{-t}{1+t}\right)^{\frac{2}{3}} dt \cdot e^{i\frac{2\pi}{3}}$$

$$= \frac{2\pi i}{3} + \int_0^1 f(t) \left(\frac{t}{1-t}\right)^{\frac{2}{3}} dt \cdot e^{-\frac{2i\pi}{3}} - \underbrace{\int_{-1}^{0} f(-t) \left(\frac{-t}{1+t}\right)^{\frac{2}{3}} dt \cdot e^{i\frac{2\pi}{3}}}_{u=-t,\ du=-dt}$$

$$'' \int_1^0 f(u) \left(\frac{u}{1-u}\right)^{\frac{2}{3}} (-du)\, e^{i\frac{2\pi}{3}}$$

$$= \frac{2\pi i}{3} + \left(-e^{\frac{2\pi i}{3}} + e^{-\frac{2i\pi}{3}}\right) I$$

$$= \frac{2\pi i}{3} - 2i \underbrace{\sin\frac{2\pi}{3}}_{\frac{\sqrt{3}}{2}} I = \frac{2i\pi}{3} - i\sqrt{3} I \Rightarrow I = \left(\frac{2i\pi}{3} - 2i\pi \frac{\sqrt{3}-1}{2\sqrt[3]{2}}\right) \cdot \frac{1}{i\sqrt{3}}$$

$$I = \frac{2\pi}{\sqrt{3}} \left(\frac{1}{3} - \frac{\sqrt{3}-1}{2\sqrt[3]{2}}\right)$$