

# Mechanics 2/23

Lecture notes, 2010/11, teaching block 1



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# Chapter 0

## Introduction

In this course we are going to consider a formulation of mechanics (variational mechanics) that is different from the one in Mechanics 1 (Newtonian mechanics). I first want to explain why such a formulation is called for, and discuss some of the main ideas.

### Newtonian mechanics

The central result in Newtonian mechanics concerns the acceleration of a particle, i.e., the second derivative of its position  $\mathbf{r}$  w.r.t. time. This acceleration is given by the ratio of the force  $\mathbf{F}$  acting on the particle and its mass  $m$ , i.e.,

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\mathbf{F}}{m}.$$

The force  $\mathbf{F}$  typically depends on the particle position  $\mathbf{r}$ ; in addition it may depend on the positions of other particles and on the velocities of the particles involved. Thus we obtain differential equations that contains both particle positions and their second (and sometimes its first) derivatives, i.e., **second-order ordinary differential equations**.

These differential equations provide a way of determining the trajectories of particles. However, their direct application can sometimes become messy. Historically, the first applications where this was felt in engineering and in astronomy.

### Variational mechanics

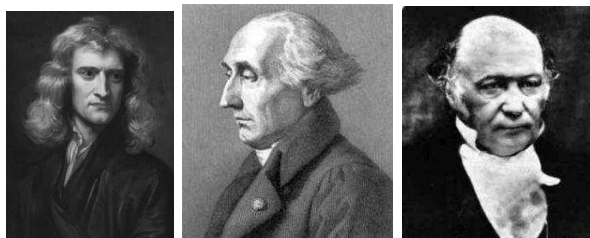


Figure 1: Isaac Newton, Joseph Louis Lagrange, and William Rowan Hamilton.

To simplify the treatment of such complex problems, a variational formulation of mechanics was developed. There are two different versions of variational me-

chanics, respectively introduced by Joseph Louis Lagrange (1736-1813) and William Rowan Hamilton (1805-1865). To explain the key idea of these approaches, let us consider the simplest case possible: one particle on which no forces are acting. We then have  $\frac{d^2\mathbf{r}}{dt^2} = 0$ , i.e., the particle travels with constant velocity  $\frac{d\mathbf{r}}{dt}$  along a straight line. Now the important point is that the **straight line** is also the **shortest connection** between two points. Thus, instead of using a differential equation, we would have obtained a trajectory of the same form if we had postulated that without forces particles always follow the shortest line connecting two points.

The main idea of variational mechanics is to generalise this simple observation to more general settings. Our goal is thus to formulate mechanics through a variational principle: **Particles travel on trajectories that extremise (minimise, maximise) “something”**. This “something” is not always the length, and one of our goals will be to find what it is. In general it will be called the “action”.

### Advantages of variational mechanics

The main advantages of the variational approach to mechanics are:

- In mechanics it is often helpful to work in **coordinates** adapted to the system we are interested in. For example, if we want to describe the motion of a satellite in the gravitational field of the Earth, it is helpful to use spherical coordinates with the centre of the Earth as the origin. More complicated systems require other complicated sets of coordinates. For example if we consider a particle moving in the gravitational field of two masses it would be better to use elliptic coordinates (see Fig. 2). In these coordinates two points are singled out, corresponding to the positions of the two masses.

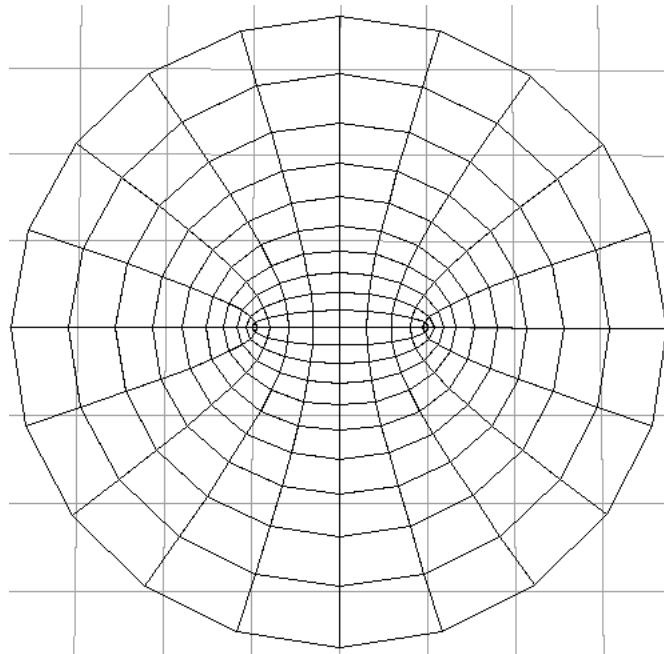


Figure 2: Elliptic coordinates.

We shall see that in variational mechanics it becomes particularly simple to switch between coordinate systems.

- Many mechanical systems have **constraints**, i.e., conditions where particles or bodies are allowed to be and where not. A practical example would be a train that is required to stay on the railroad track. In variational mechanics these constraints can easily be incorporated by choosing appropriate coordinates, e.g., by taking the railroad track as a coordinate line.
- Crucially, many areas of **mathematical physics** are formulated in the language developed in variational mechanics. For instance “Hamiltonians” and “Lagrangians” play an important role in quantum mechanics or chaos theory. Similarly the ideas underlying variational mechanics are important in the theory of **differential equations**.

### (Rough) outline of this course

In this course, we will first develop the mathematical tools needed for extremisation problems like the one sketched above (**variational calculus**). We will then consider both the **Lagrangian** and the **Hamiltonian** formulation of mechanics and several examples for their use.

### Reminder: Chain rule

As a preparation for many calculations in this course I want to remind how derivatives of expressions like

$$f(u(t), v(t), t)$$

are evaluated. According to the chain rule, we first have to differentiate w.r.t. the arguments of  $f$ , and then multiply with the derivatives of these arguments w.r.t.  $t$ . This yields

$$\frac{d}{dt}f(u(t), v(t), t) = \frac{\partial f}{\partial u}(u(t), v(t), t)\frac{du}{dt} + \frac{\partial f}{\partial v}(u(t), v(t), t)\frac{dv}{dt} + \frac{\partial f}{\partial t}(u(t), v(t), t) .$$

Dropping the arguments and setting  $\dot{u} = \frac{du}{dt}$ ,  $\dot{v} = \frac{dv}{dt}$ , we can also write

$$\frac{df}{dt} = \frac{\partial f}{\partial u}\dot{u} + \frac{\partial f}{\partial v}\dot{v} + \frac{\partial f}{\partial t} .$$

It is important to distinguish between the **partial derivative**  $\frac{\partial f}{\partial t}$  with respect to the third argument of  $f$ , and the **total derivative**  $\frac{df}{dt}$  where also the  $t$ -dependence of the first and the second argument are taken into account.

An example would be  $f$  representing the temperature felt by a person,  $f$  denoting the time, and  $u(t), v(t)$  the position of the person. Then the rate of change  $\frac{df}{dt}$  of the temperature felt will depend on the change of temperature with time ( $\frac{\partial f}{\partial t}$ ) and on whether the person moves to a warmer or colder place (leading to the terms  $\frac{\partial f}{\partial u}\dot{u}$  and  $\frac{\partial f}{\partial v}\dot{v}$ ).





# Chapter 1

## The calculus of variations

### 1.1 Example

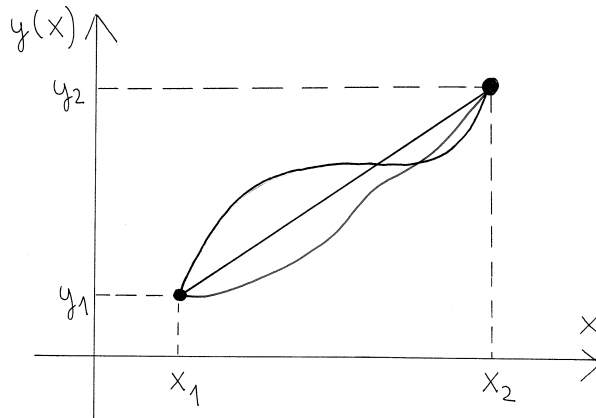


Figure 1.1: Possible connections between  $(x_1, y_1)$  and  $(x_2, y_2)$ .

To introduce variational mechanics, we first need to equip ourselves with the tools needed to solve extremisation problems. We will start with the simplest example: **Showing that the shortest connection between two points in  $\mathbb{R}^2$  is a straight line.** Let us denote these two points by coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$  in a Cartesian coordinate system, see Fig. 1.1. Curves connecting these two points can then be described through functions  $y(x)$  that assign to each  $x$ -coordinate between  $x_1$  and  $x_2$  the corresponding  $y$ -coordinate. We now have to

- determine the length  $l$  of the curve corresponding to each function  $y(x)$
- and find a  $y(x)$  such that  $l$  becomes minimal.

To start with the first task, we split the  $x$ -axis into small pieces of length  $dx$ . As seen in Fig. 1.2, this means that the curve is also split into small pieces, whose length will be denoted by  $dl$ . We assume that the pieces are small enough such that inside each piece the curve can be considered as a straight line. This means that for each piece we can draw a triangle as in Fig. 1.2 with the width  $dx$ , the height

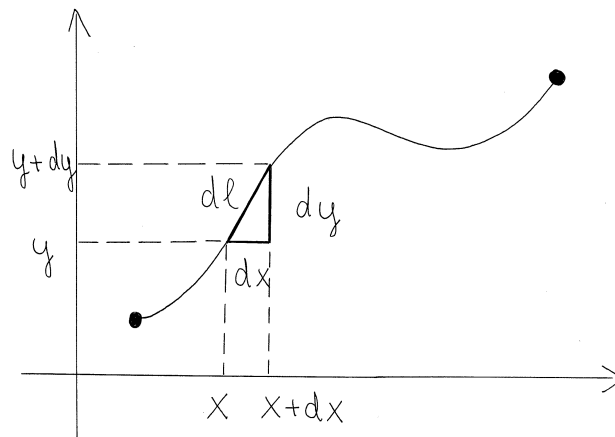


Figure 1.2: Determining the length of a curve  $y(x)$  in  $\mathbb{R}^2$ .

$dy$ , and the length  $dl$  as its sides. Since the slope of the curve must be given by the derivative  $y'(x)$ ,  $dy$  depends on  $dx$  as

$$y'(x) = \frac{dy}{dx} \Rightarrow dy = y'(x)dx .$$

We can now determine the length of each piece using Pythagoras' theorem,

$$\begin{aligned} (dl)^2 &= (dx)^2 + (dy)^2 = (1 + y'(x)^2)(dx)^2 \\ \Rightarrow dl &= \sqrt{1 + y'(x)^2}dx . \end{aligned}$$

To obtain the overall length of the curve, we have to sum over the lengths  $dl$  of all pieces. If we take the limit where the width  $dx$  of the pieces goes to zero, this sum can be replaced by the integral, and we obtain the

**Length of a curve in  $\mathbb{R}^2$**

$$l = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}dx .$$

Our task is now to **find functions  $y(x)$  with  $y(x_1) = y_1$  and  $y(x_2) = y_2$  for which the length  $l$  becomes minimal.**

## 1.2 Generalisation

The problem outlined above is the simplest example for a more general type of variational problems, where we maximise, minimise, or in general look for stationary points of an integral of a function depending on  $y(x)$ ,  $y'(x)$  and  $x$ :

### Stationarity problem

Let

$$K[y] = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx . \quad (1.1)$$

Find  $y(x)$  satisfying the boundary conditions  $y(x_1) = y_1$ ,  $y(x_2) = y_2$  for which  $K[y]$  becomes stationary.

The stationarity problem considered here are is different from those in Calculus, where the quantity to be minimised depended only on a single number or on a vector. In contrast  $K[y]$  depends on a function. The mapping from  $y$  to  $K[y]$  is an example of a **functional**. A functional is a function that maps functions to real numbers. Not all functionals are of the form in Eq. (1.1) (e.g. we might also have integrals depending on  $y''(x)$ ), but those of the type in Eq. (1.1) play a particularly important role in mechanics. Another example for such a  $K[y]$  would be  $K[y] = \int_{x_1}^{x_2} (y(x)^2 + y'(x)^2)e^{-x} dx$ .

To proceed, we have to understand better what it means for a functional to be **stationary**. We will define stationarity of functionals in terms of a stationarity problem we are already familiar with: finding stationary points of a function that only depends on one variable. As a preparation let us consider a function  $y_*(x)$  satisfying our boundary conditions  $y_*(x_1) = y_1$ ,  $y_*(x_2) = y_2$  and then look at functions of the type

$$y_*(x) + ah(x) .$$

Here  $a$  is a real number, and  $h(x)$  is an arbitrary but fixed function that satisfies the boundary conditions  $h(x_1) = h(x_2) = 0$ . The functions  $y_*(x) + ah(x)$  then satisfy the same boundary conditions as  $y_*(x)$ , i.e.,  $y_*(x_1) + ah(x_1) = y_1$ ,  $y_*(x_2) + ah(x_2) = y_2$ . We can evaluate the functional  $K$  with these functions as parameters, and get

$$K[y_* + ah] = \int_{x_1}^{x_2} f(y_*(x) + ah(x), y_*'(x) + ah'(x), x) dx \quad (1.2)$$

which is just a real number. We can now consider the values of  $K[y_* + ah]$  we get for different  $a$ , and look for which values of  $a$  our  $K[y_* + ah]$  becomes stationary. This is the type of stationarity problem we are know from calculus.  $K[y_* + ah]$  becomes stationary for all  $a$  with

$$\frac{d}{da} K[y_* + ah] = 0 . \quad (1.3)$$

Now the point  $a = 0$  corresponds to the function  $y_*$ , since for  $a = 0$  we have  $y_* + ah = y_*$ . For  $K$  to be stationary at  $y_*$  we thus have to demand that (1.3) holds for  $a = 0$ . Since  $h$  was arbitrary, we have to demand this for all admissible functions  $h$ . We thus obtain the following definition of stationarity for functionals:

$K$  is **stationary** at a function  $y_*$  if

$$\frac{d}{da} K[y_* + ah]|_{a=0} = 0 \quad (1.4)$$

holds for all functions  $h(x)$  with  $h(x_1) = h(x_2) = 0$ .

### 1.3 Euler-Lagrange equation

We now want to derive a differential equation that can be used to find functions  $y_*(x)$  for which  $K$  becomes stationary. To do so, let us assume that the function  $y_*(x)$  is such a stationary point, and that it is also differentiable and satisfies our boundary conditions.

**First step: Use (1.4)**

If we use (1.2) and the chain rule, Eq. (1.4) turns into

$$\begin{aligned} 0 &= \frac{d}{da} K[y_* + ah] \Big|_{a=0} \\ &= \int_{x_1}^{x_2} \left\{ \frac{d}{da} f(y_* + ah, y_*' + ah', x) \right\}_{a=0} dx \\ &= \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y}(y_* + ah, y_*' + ah', x)h(x) + \frac{\partial f}{\partial y'}(y_* + ah, y_*' + ah', x)h'(x) \right\}_{a=0} dx. \end{aligned}$$

We thus obtain

$$0 = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y}(y_*(x), y_*'(x), x)h(x) + \frac{\partial f}{\partial y'}(y_*(x), y_*'(x), x)h'(x) \right\} dx \quad (1.5)$$

which needs to hold for all  $h(x)$  vanishing at  $x_1$  and  $x_2$ . (At this point, it is also convenient to drop the stars of  $y_*$ .)

**Second step: Integrate by parts**

We would like to simplify (1.5) such that both summands become proportional to  $h(x)$ . This is easily achieved if we take the second summand and integrate by parts,

$$\begin{aligned} & \int_{x_1}^{x_2} \underbrace{\frac{\partial f}{\partial y'}}_u \underbrace{h'}_{v'} dx \\ &= \frac{\partial f}{\partial y'} h \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) h dx \\ &= - \int_{x_1}^{x_2} \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) h dx. \end{aligned}$$

Here the term  $\frac{\partial f}{\partial y'} h \Big|_{x_1}^{x_2}$  could be dropped due to the boundary condition  $h(x_1) = h(x_2) = 0$ . Inserting this result into (1.5) we obtain

$$\int_{x_1}^{x_2} \left( \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) \right) h(x) = 0 \quad (1.6)$$

for arbitrary  $h(x)$ .

**Third step: Fundamental lemma of variational calculus**

Obviously, Eq. (1.6) is satisfied if the term in brackets vanishes. Indeed the fundamental lemma of variational calculus guarantees that this is the only way to satisfy the equation.

**Fundamental lemma of variational calculus**

Suppose that  $g(x)$  is continuous, and

$$\int_{x_1}^{x_2} g(x)h(x) = 0 \quad (1.7)$$

holds for all  $h(x)$  satisfying  $h(x_1) = h(x_2) = 0$ . Then  $g(x) = 0$  for all  $x \in (x_1, x_2)$ .

**Proof:** The proof proceeds by contradiction. Suppose that Eq. (1.7) holds for all  $h(x)$  and that we nevertheless have  $g(b) \neq 0$  for one  $b \in (x_1, x_2)$ , say,  $g(b) > 0$  (see Fig. 1.3). Due to the continuity of  $g(x)$  this implies that we have  $g(x) > 0$  in an interval around  $b$ . We now pick  $h(x)$  such that  $h(x) > 0$  in this interval and  $h(x) = 0$  outside. Then the integrand in Eq. (1.7) is positive in our interval and zero outside, and the integral must be positive as well. This contradicts our assumption. Hence the theorem is proven.

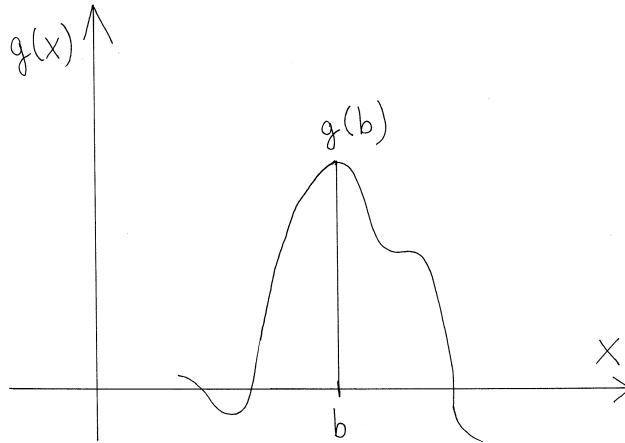


Figure 1.3: The continuous function  $g(x)$  is positive in an interval around  $b$ .

The fundamental lemma of variational calculus implies that the term in brackets in Eq. (1.6) vanishes. We thus see that functions extremising  $K = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx$  have to satisfy the

**Euler-Lagrange equation**

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0$$

If we compare the Euler-Lagrange equation to the stationarity condition for functions of a single variable, it appears natural to identify the term of the left-hand side with a kind of derivative. Indeed,

$$\frac{\delta K}{\delta y} = \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'}$$

is known as the **functional derivative** of  $K$ , and one can build a whole theory of **differentiation with respect to functions** largely analogous to differentiation with respect to numbers or vectors.<sup>1</sup> This theory will not be considered further in this lecture, since the Euler-Lagrange equation is already sufficient for our purposes.

## 1.4 Solution of our problem

The stationarity problem above was initially motivated by the search for curves where the length

$$l = \int_{x_1}^{x_2} (1 + y'(x)^2)^{1/2} dx$$

becomes minimal. We are now ready to solve this problem. The Euler-Lagrange equation for  $f(y, y', x) = (1 + y'^2)^{1/2}$  reads

$$\frac{\partial(1 + y'^2)^{1/2}}{\partial y} - \frac{d}{dx} \frac{\partial(1 + y'^2)^{1/2}}{\partial y'} = 0.$$

The summand on the left-hand side vanishes since  $(1 + y'^2)^{1/2}$  is independent of  $y$ . We thus find

$$\begin{aligned} \frac{d}{dx} \left[ (1 + y'^2)^{-1/2} y' \right] &= 0 \\ \Rightarrow (1 + y'^2)^{-1/2} y' &= \text{const} \\ \Rightarrow y' &= \text{const}. \end{aligned}$$

The only curves with constant derivatives  $y'$  are straight lines

$$y(x) = y'x + b.$$

The constants  $y', b$  are now determined by the boundary conditions  $y(x_1) = y_1, y(x_2) = y_2$ . To incorporate  $y(x_1) = y_1$  we set  $b = y_1 - y' \cdot x_1$ . This yields

$$y(x) = y' \cdot (x - x_1) + y_1$$

and indeed  $y(x_1) = y_1$ . The constant derivative  $y'$  must then coincide with the slope of the straight line leading from  $(x_1, y_1)$  to  $(x_2, y_2)$

$$y' = \frac{y_2 - y_1}{x_2 - x_1}.$$

## 1.5 Alternative version of the Euler-Lagrange equation

If the integrand  $f$  in  $K$  does not depend explicitly on  $x$  the Euler-Lagrange equation can be brought to a simpler form, also known as the Beltrami equation.

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<sup>1</sup>See, e.g., Arfken, *Mathematical Methods for Physicists* or Gelfand and Fomin, *Calculus of Variations*.

### Alternative version of the Euler-Lagrange equation

If  $f = f(y, y')$  the Euler-Lagrange equation turns into

$$f - \frac{\partial f}{\partial y'} y' = \text{const}$$

(where "const" means that  $f - \frac{\partial f}{\partial y'} y'$  does not depend on  $x$ ).

**Proof:** Just use the chain rule to compute the total derivative of  $f - \frac{\partial f}{\partial y'} y'$  w.r.t.  $x$ :

$$\frac{d}{dx} \left( f - \frac{\partial f}{\partial y'} y' \right) = \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} - \left( \frac{d}{dx} \frac{\partial f}{\partial y'} \right) y' - \frac{\partial f}{\partial y'} \frac{dy'}{dx}$$

Here the second and the fourth summand cancel. If we use the Euler-Lagrange equation  $\frac{d}{dx} \frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}$  we see that also the first and third terms cancel, and we have

$$\frac{d}{dx} \left( f - \frac{\partial f}{\partial y'} y' \right) = 0,$$

as desired.

**Note:** In contrast to the original Euler-Lagrange equation, this is a first-order differential equation. It is typically much easier to use, since we can save the labour of taking derivatives w.r.t.  $y$  and  $x$ .

If we apply this formula our problem of finding the curve  $y(x)$  with minimal length  $l = \int_{x_1}^{x_2} (1 + y'(x)^2)^{1/2}$ , we find

$$(1 + y'^2)^{1/2} - (1 + y'^2)^{-1/2} y'^2 = \text{const} \Rightarrow y' = \text{const}$$

as before, i.e., we again see that the shortest connection between two points is a straight line.

## 1.6 The brachistochrone



Figure 1.4: Johann Bernoulli, Isaac Newton, Gottfried Leibniz, Guillaume de l'Hopital, and Jakob Bernoulli.

A classical problem in the calculus of variations in the brachistochrone problem (from Greek  $\beta\rho\chi\sigma\iota\zeta\tau\omicron\varsigma$   $\chi\rho\omicron\nu\omicron\varsigma$ , which means "shortest time"). The brachistochrone problem was posed by Johann Bernoulli (1667-1746) in *Acta Eruditorum* in 1696. He introduced the problem as follows:-

I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.

Solutions were given by Isaac Newton (1643-1727), Gottfried Leibniz (1646-1716), Guillaume de l'Hopital (1661-1704) and Jakob Bernoulli (1654-1705, brother of the above).

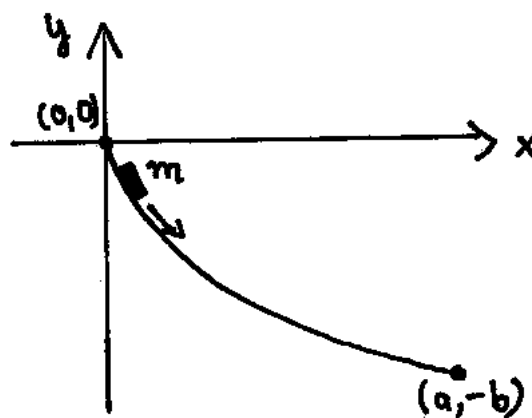


Figure 1.5: The brachistochrone problem.

The problem is to find the ideal form of a slide, such that a mass  $m$  that is initially at rest at the origin  $(0,0)$  can be brought down to a point  $(a,-b)$ ,  $b > 0$  in the shortest time possible, see Fig. 1.5. It is assumed that the only force acting on the mass is gravity; there is no friction. The optimal form of the slide should then be represented as a function  $y(x)$  that assigns to each  $x$  coordinate the corresponding height  $y$ .

Among all possible paths between  $(0,0)$  and  $(a,-b)$  the straight line would have the shortest length. On the other hand, the mass is the faster the lower it is since then more of its potential energy has been converted into kinetic energy. We thus expect that a slide where the mass goes down quickly (e.g. from  $(0,0)$  to  $(0,-b)$ ) and then to  $(a,-b)$ ) will be a good choice as well. One might guess that the optimal solution should lie somewhere in between these possibilities.

#### Find time $t$ for given $y(x)$

To solve the problem, we first have to find the time  $t$  the mass needs to arrive at  $(a,-b)$  as a functional of the curve  $y(x)$ . To get this time, we again split the curve into pieces. As we have seen in Section 2.1, the length of each piece is given by  $dl = (1 + y'(x)^2)^{1/2} dx$ . The time  $dt$  the mass spends in each piece is then given by  $dt = \frac{dl}{v}$  where  $v$  is the speed of the mass. This speed can be inferred from energy



conservation. At the starting point  $(0, 0)$  the mass neither has potential nor kinetic energy. At a later point at height  $y < 0$  the potential energy is  $mgy$ , and the kinetic energy has increased to  $\frac{m}{2}v^2$ . Energy conservation now implies that

$$\begin{aligned} E = 0 &= mgy + \frac{1}{2}mv^2 \\ \Rightarrow v &= (-2gy)^{1/2}. \end{aligned}$$

Using this result for the speed and the length  $dl$  calculated above we obtain the time spend in each piece of the curve as

$$dt = \frac{dl}{v} = \left( \frac{1 + y'(x)^2}{-2gy(x)} \right)^{1/2}.$$

If we integrate over  $dt$  we see that the travel time of the mass is given by

$$t = \frac{1}{(2g)^{1/2}} \int_0^a \underbrace{\left( \frac{1 + y'(x)^2}{-y(x)} \right)^{1/2}}_{\equiv f} dx.$$

### Find minima

We now have to find the curve  $y(x)$  for which  $t$  becomes minimal. To do so we use the **alternative version of the Euler-Lagrange equation**

$$f - \frac{\partial f}{\partial y'} y' = \text{const}$$

If we insert the  $f$  given above, we obtain

$$\begin{aligned} \left( \frac{1 + y'^2}{-y} \right)^{1/2} - \frac{\frac{1}{2}(1 + y'^2)^{-1/2} 2y'}{(-y)^{1/2}} y' &= \text{const} \\ \Rightarrow \frac{1}{(-y)^{1/2}(1 + y'^2)^{1/2}} &= \text{const} \end{aligned}$$

Thus the term  $(-y)(1 + y'^2)$  in the denominator must be constant. If we denote this constant by  $2R$  we obtain

$$\begin{aligned} 1 + y'^2 &= \frac{2R}{-y} \\ \Rightarrow y' &= \mp \left( \frac{2R + y}{-y} \right)^{1/2} \end{aligned} \quad (1.8)$$

At least initially, the sign above must be negative since we would expect the mass to fall down. We have thus obtained a differential equation giving the derivative  $y'$  as a function of  $y$ .

### Solving the differential equation

We solve the differential equation by separation. We thus write  $y' = \frac{dy}{dx} \Rightarrow \frac{dy}{y'} = dx$  and then integrate on both sides. In our case the integration limits have to be 0 and  $a$  for  $x$  and  $y(0) = 0$  and  $y(x)$  for  $y$ . We thus obtain

$$\int_0^{y(x)} \frac{dy}{y'} = \int_0^x d\tilde{x} = x$$

and, inserting the formula for  $y'$ ,

$$-\int_0^{y(x)} \left( \frac{-y}{2R+y} \right)^{1/2} dy = x. \quad (1.9)$$

To proceed we need to evaluate the integral over  $y$ . This can be accomplished with the trigonometric substitution

$$y = -2R \sin^2 \frac{\theta}{2} \quad (1.10)$$

Note that in the beginning of the curve we must have  $y = 0$  and thus  $\theta = 0$ . Differentiation now yields

$$dy = -2R \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta$$

and the denominator in Eq. (1.9) simplifies to

$$(2R+y)^{1/2} = (2R(1 - \sin^2 \frac{\theta}{2}))^{1/2} = (2R)^{1/2} \cos \frac{\theta}{2}$$

If we substitute all these formulas into Eq. (1.9) we obtain

$$x = 2R \int_0^{\theta(x)} \sin^2 \frac{\theta}{2} d\theta = R \int_0^{\theta(x)} (1 - \cos \theta) d\theta = R(\theta(x) - \sin \theta(x)). \quad (1.11)$$

### Result

One could attempt to solve (1.10) and (1.11) to get  $y$  in terms of  $x$  but it is easier to represent the solution curve in parametrised form, leaving both  $x$  and  $y$  as functions of  $\theta$ . Indeed (1.10) and (1.11) boil down to

$$\begin{aligned} x(\theta) &= R(\theta - \sin \theta) \\ y(\theta) &= -2R \sin^2 \frac{\theta}{2} = -R(1 - \cos \theta) \end{aligned} \quad (1.12)$$

A plot of the resulting  $x(\theta)$ ,  $y(\theta)$  is shown in figure 1.6. We see that for  $\theta = 0, 2\pi, 4\pi, \dots$  the coordinate  $y$  reaches zero whereas  $x$  is equal to  $0, 2\pi R, 4\pi R, \dots$ . At these points the curve has a cusp, and the slope becomes infinite on both sides of the cusp. At  $\theta = \pi, 3\pi, 5\pi, \dots$  there are minima with  $y = -2R$ . Strictly speaking our results only apply up to the first minimum at  $\theta = \pi$  since we only considered the regime where  $y$  decreases with increasing  $x$  and chose the sign in (1.8) accordingly. However by repeating our calculation with small modifications for other values of  $\theta$ , one can see that the solution obtained is actually valid for arbitrary  $\theta$ .

It is instructive to write (1.12) in vector notation,

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ -R \end{pmatrix} + \begin{pmatrix} R \\ 0 \end{pmatrix} \theta + R \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \quad (1.13)$$

Eq. (1.13) has the following interpretation: If we momentarily forget about the third term on the right-hand side of (1.13), the vectors  $(x, y)$  would start from  $(0, -R)$  at  $\theta = 0$ ; as  $\theta$  increases they would then move on a straight line in positive  $x$ -direction. In contrast, the third term describes a motion around a circle of radius  $R$ , i.e., a rotation. We thus see that as  $\theta$  increases the points  $(x, y)$  follow a superposition of a motion on a straight line and a rotation.

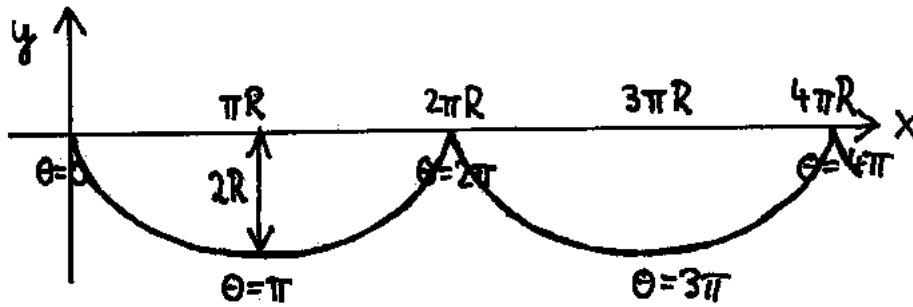


Figure 1.6:  $\theta$ ,  $x$ , and  $y$  for the brachistochrone problem.

### A related problem

The same curve (1.13) but with opposite sign of  $y$  is obtained in a completely different problem: the motion of a disk with radius  $R$  rolling over the  $x$ -axis. This curve is called the **cyloid** curve.

To also get the sign as in (1.13) we instead compare with the more artificial situation where a disk is “rolling” *below* the  $x$ -axis. If the disk is rolling into positive  $x$  direction, its centre point moves on a straight line with increasing  $x$ . A point  $(x, y)$  on the circumference of the disk follows that straight-line motion, but at the same time rotates around the centre. If one observes a rolling disk, one can see that the point actually rotates to the left. This is in line with (1.13) since  $(-\sin \theta, \cos \theta) = (\cos(\frac{\pi}{2} + \theta), \sin(\frac{\pi}{2} + \theta))$  gives a rotation to the left if  $\theta$  is increased. During one revolution (i.e. an increase of  $\theta$  by  $2\pi$ ) the disk should move by a distance that coincides with the circumference  $2\pi R$ , i.e.,  $x$  should increase by  $2\pi R$  just as in (1.13).

### Boundary conditions

We still have to find the right value for  $R$ . Moreover we must find out which part of Fig. 1.6 should be taken as our ideal slide; this part must certainly start at the origin but we have to know at which value  $\theta_{\text{end}}$  of  $\theta$  it has to end.  $R$  and  $\theta_{\text{end}}$  can be found by inserting the boundary conditions  $x(\theta_{\text{end}}) = a$ ,  $y(\theta_{\text{end}}) = -b$  into Eq. (1.13). We then obtain the nonlinear system of equations

$$\begin{aligned} a &= R(\theta_{\text{end}} - \sin \theta_{\text{end}}) \\ -b &= -R(1 - \cos \theta_{\text{end}}) \end{aligned}$$

which will be considered further on the second problem sheet.

Qualitatively the most interesting question is whether the slide will end to the left or to the right of the minimum at  $\theta = \pi$ . In the first case the slide will always go down (see Fig. 1.6a) whereas in the second case it will first go down and then up again (see Fig. 1.6b). If we compare the curve in Fig. 1.6 to the line  $y = -\frac{2x}{\pi}$ , we see that to the left of the minimum the curve always satisfies  $y < -\frac{2x}{\pi}$  whereas to the right we have  $y > -\frac{2x}{\pi}$  inserting  $x(\theta_{\text{end}}) = a$ ,  $y(\theta_{\text{end}}) = -b$  we thus see that the endpoint will be to the left if

$$-b < -\frac{2a}{\pi} \Leftrightarrow \frac{2a}{b} < \pi. \quad (1.14)$$

In this case the slide will always go down whereas otherwise it first goes down and then up.

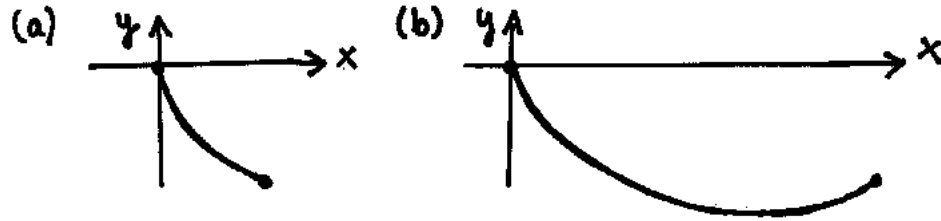


Figure 1.7: Solution of the brachistochrone problem: (a) for  $\frac{2a}{b} < \pi$ , (b) for  $\frac{2a}{b} > \pi$ .

## 1.7 Functionals depending on several functions

So far we only considered variational problems that involved one function  $y(x)$  mapping  $\mathbb{R}$  to  $\mathbb{R}$ . However one often encounters problems that involve several functions of this type. Let us thus consider  $n$  different functions  $y_1(x), y_2(x), \dots, y_n(x)$ , each subject to boundary conditions at  $x_1$  and  $x_2$ , and then define a functional depending on all of them

$$K[y_1, \dots, y_n] = \int_{x_1}^{x_2} f(y_1(x), \dots, y_n(x), y_1'(x), \dots, y_n'(x), x) dx .$$

We want to find  $y_1(x), y_2(x), \dots, y_n(x)$  such that  $K$  becomes stationary w.r.t. variations of all  $n$  functions.  $K$  will be stationary w.r.t. variations of  $y_j(x)$  if the corresponding Euler-Lagrange equation

$$\frac{\partial f}{\partial y_j} - \frac{d}{dx} \frac{\partial f}{\partial y_j'} = 0 \quad (1.15)$$

is satisfied. To make  $K$  stationary w.r.t. variations of all function we thus have to demand that (1.15) holds **for all  $j$  from 1 to  $n$** .

**Note:** It is often convenient to collect all functions  $y_j(x)$  into a vector-valued function  $\mathbf{y}(x) = (y_1(x), \dots, y_n(x))$ .

## 1.8 Fermat's principle

To illustrate the use of variational calculus in  $\mathbb{R}^n$  we consider an example from optics. (Geometric) optics can be based on a variational principle formulated by Fermat:

### Fermat's principle

Light travels between two points on paths (rays) that take the least (or stationary) time.

These light rays are very similar to particle trajectories in mechanics. To make use of Fermat's principle we have to recall that the speed of light is given by  $\frac{c}{n}$  where

$c$  is the speed of light in vacuum and the **refraction index**  $n > 1$  depends on the medium in which the light is propagating. In many applications the refraction index and thus the speed of light is constant. Then Fermat's principle implies that also the **length** of the rays is stationary. Light thus moves on straight lines. (It can also be reflected by a mirror; as seen on a problem sheet rays where light is reflected from the mirror according to the reflection law "angle of incidence = angle of reflection" have stationary length.)

We now want to consider the situation where the **refraction index is not constant**, i.e., we have  $n = n(x, y, z)$ .

**Find  $t$  as functional of the path**

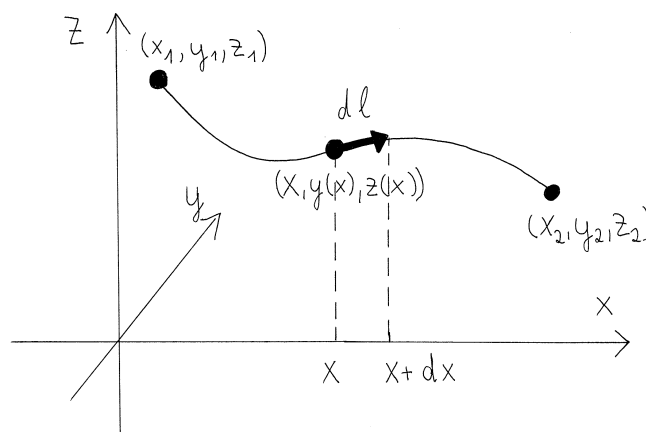


Figure 1.8: Propagation of a light ray in  $\mathbb{R}^3$ .

We first need to get the travel time  $t$  as a functional of the path. As shown in figure 1.8 the path can be described by functions  $y(x)$  and  $z(x)$  that assign to each value of  $x$  the corresponding coordinates  $y$  and  $z$ . If we break the path into pieces, reasoning analogous to section 1.1 and 1.6 shows that the length  $dl$  of each piece is given by

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

Using that  $dy = y'dx$ ,  $dz = z'dx$  we thus find

$$dl = (1 + y'^2 + z'^2)^{1/2} dx .$$

The travel time corresponding to  $dl$  can be obtained if we divide by the velocity of the path  $\frac{c}{n}$ . We then obtain

$$dt = \frac{n}{c} dl$$

and integration yields the travel time of the whole ray

$$t = \int_{x_1}^{x_2} \frac{n(x, y(x), z(x))}{c} (1 + y'(x)^2 + z'(x)^2)^{1/2} dx$$

**Find stationary points of  $t$** 

We now need to find  $y(x)$ ,  $z(x)$  such that the travel time  $t$  becomes extremal. We thus face an extremisation problem of the type discussed in section 1.7, with the identifications

$$\begin{aligned} K &= t \\ (y_1, y_2) &= (y, z) \\ f &= \frac{n(x, y(x), z(x))}{c} \underbrace{(1 + y'(x)^2 + z'(x)^2)^{1/2}}_{=u(x)} \end{aligned}$$

To solve this extremisation problem we have to consider the **Euler-Lagrange equations** for both  $y(x)$  and  $z(x)$ . For  $y(x)$  we obtain

$$\begin{aligned} \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} &= 0 \\ \Rightarrow \frac{1}{c} \frac{\partial n}{\partial y} u - \frac{d}{dx} \left( \frac{n}{c} \frac{1}{2} u^{-1} 2y' \right) &= 0 \\ \Rightarrow \frac{d}{dx} \left( \frac{ny'}{u} \right) &= \frac{\partial n}{\partial y} u \end{aligned} \tag{1.16}$$

For  $z(x)$  analogous reasoning yields

$$\begin{aligned} \frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z'} &= 0 \\ \Rightarrow \frac{d}{dx} \left( \frac{nz'}{u} \right) &= \frac{\partial n}{\partial z} u \end{aligned} \tag{1.17}$$

These equations must now be solved for  $y(x)$ ,  $z(x)$  to get rays.

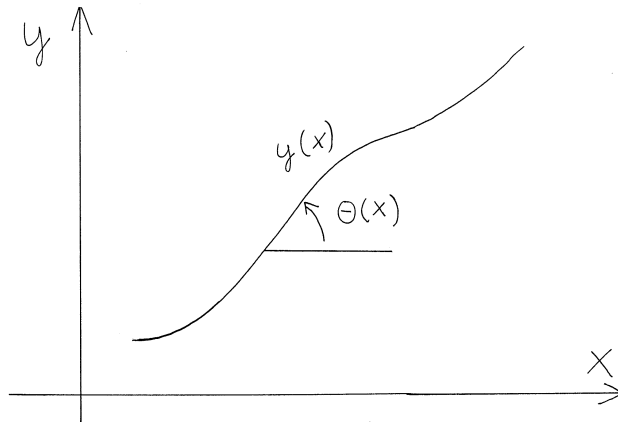
**Special case**

Figure 1.9: A ray of light in two dimensions.

For definiteness, let us consider the special case where the **refraction index depends only on  $x$** , i.e.  $n = n(x)$  and  $\frac{\partial n}{\partial y} = \frac{\partial n}{\partial z} = 0$ . Then the Euler-Lagrange equations (1.16) and (1.17) boil down to

$$\begin{aligned}\frac{ny'}{u} &= \text{const} \\ \frac{nz'}{u} &= \text{const}\end{aligned}\tag{1.18}$$

Let us moreover assume that the  $z$ -coordinate is always zero, a condition that is certainly in line with (1.18). In this case the light rays only travel in the  $x$ - $y$ -plane, see figure 1.9. If we insert the definition of  $u$  the second equation in (1.18) then boils down to

$$\frac{ny'}{(1+y'^2)^{1/2}} = \text{const}\tag{1.19}$$

Eq. (1.19) can be simplified if we express the slope  $y'(x)$  of the curve in Fig. 1.9 through the angle  $\theta(x)$  enclosed between the curve and the  $x$ -direction. We then write

$$y'(x) = \tan \theta(x)$$

and simplify the denominator in (1.19),

$$(1+y'(x)^2)^{1/2} = (1+\tan^2 \theta(x))^{1/2} = \left(\frac{\cos^2 \theta(x) + \sin^2 \theta(x)}{\cos^2 \theta(x)}\right)^{1/2} = \frac{1}{\cos \theta(x)}.$$

Eq. (1.19) thus turns into

$$n(x) \sin \theta(x) = \text{const}$$

which is known as (the generalised version of) **Snell's law**.

For example, let us assume that the space  $x < 0$  is filled with a medium with constant refraction index  $n = n_1$  (say, air) and the space  $x > 0$  is filled with a different medium with  $n = n_2$  (say, glass), see Fig. 1.10. Inside both media the rays travel on straight lines enclosing angles  $\theta_1$  and  $\theta_2$  with the  $x$ -direction. Given  $\theta_1$  the angle  $\theta_2$  is then determined by

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

This equation is the original formulation of Snell's law.

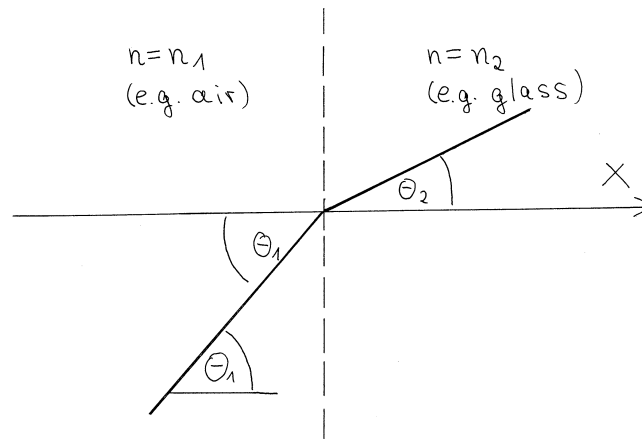


Figure 1.10: Snell's law.



## Chapter 2

# Lagrangian mechanics

### 2.1 Reminder: Newton

We now want to formulate mechanics through a variational principle. As a preparation, we need to review some basic facts from Newtonian mechanics. We consider systems of  $N$  particles with masses  $m_1, m_2, \dots, m_N$  at positions  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N$ . Each of these positions is a vector  $\mathbf{r}_i = (x_i, y_i, z_i) \in \mathbb{R}^3$ .

The particle trajectories  $\mathbf{r}_i(t)$  are now determined by **Newton's second law**

$$m_i \ddot{\mathbf{r}}_i(t) = \mathbf{F}_i(\mathbf{r}_1(t), \dots, \mathbf{r}_N(t), \dot{\mathbf{r}}_1(t), \dots, \dot{\mathbf{r}}_N(t))$$

where  $\mathbf{F}_i$  is the **force** acting on the  $i$ -th particle. A particularly important type of forces are **conservative** forces: Forces are conservative if they can be written as derivatives of a **potential energy**  $U$ ,

$$\mathbf{F}_i = -\frac{\partial}{\partial \mathbf{r}_i} U(\mathbf{r}_1, \dots, \mathbf{r}_N).$$

Here  $\frac{\partial U}{\partial \mathbf{r}_i}$  is the gradient  $\begin{pmatrix} \partial U / \partial x_i \\ \partial U / \partial y_i \\ \partial U / \partial z_i \end{pmatrix}$ .

**Example:** In a **uniform gravity field** (e.g. the gravity field of the earth in the vicinity of the surface of the earth) a particle at  $\mathbf{r} = (x, y, z)$  has the potential

$$U = mgz,$$

and thus feels the force

$$\mathbf{F} = -\begin{pmatrix} \partial U / \partial x \\ \partial U / \partial y \\ \partial U / \partial z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

Finally, our system of  $N$  particles has the **kinetic energy**

$$T = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2.$$

## 2.2 Lagrangian mechanics in Cartesian coordinates

In Lagrangian mechanics, the laws of motion are formulated in terms of variational calculus, i.e., by demanding that a certain functional should become stationary. This approach has several advantages over direct use of Newtonian mechanics that will become clear over the course of this lecture. We will first develop the variational formulation for the simplest case, assuming that all forces are conservative and all particle coordinates are indicated in Cartesian coordinates.

The functions to be determined in mechanics are the particle trajectories  $\mathbf{r}_i(t)$ . We thus need a variational principle that tells us how particles move from given positions  $\mathbf{r}_i(t^{(1)}) = \mathbf{r}_i^{(1)}$  at time  $t^{(1)}$  to given positions  $\mathbf{r}_i(t^{(2)}) = \mathbf{r}_i^{(2)}$  at time  $t^{(2)}$ .

In our variational principle the role of the integrand  $f$  will be played by the difference of the kinetic and potential energy, the so-called **Lagrangian**

$$L(\mathbf{r}_1, \dots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N) = T(\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N) - U(\mathbf{r}_1, \dots, \mathbf{r}_N)$$

The Lagrangian depends both on the particle positions (determining  $U$ ) and the velocities (determining  $T$ ).

We then define a functional called the **action**  $S$ , by taking the time integral of the Lagrangian from  $t^{(1)}$  to  $t^{(2)}$ :

$$S[\mathbf{r}_1, \dots, \mathbf{r}_N] = \int_{t^{(1)}}^{t^{(2)}} L(\mathbf{r}_1(t), \dots, \mathbf{r}_N(t), \dot{\mathbf{r}}_1(t), \dots, \dot{\mathbf{r}}_N(t)) dt .$$

Now the claim is that the particles travel on trajectories  $\mathbf{r}_i(t)$  for which  $S$  becomes stationary. For historical reasons this famous principle (due to Hamilton and Maupertius) is known as the principle of least action (rather than stationary action, which would be the correct terminology).

### Principle of “least” action

In a system where all forces are conservative the trajectories of particles moving from positions  $\mathbf{r}_i(t^{(1)}) = \mathbf{r}_i^{(1)}$  to positions  $\mathbf{r}_i(t^{(2)}) = \mathbf{r}_i^{(2)}$  are chosen such that the action becomes stationary w.r.t. variations that preserve these boundary conditions, i.e., the Euler-Lagrange equations

$$\frac{\partial L}{\partial \mathbf{r}_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \quad (2.1)$$

must be satisfied, for all  $i = 1 \dots N$ .

Here the vector equation (2.1) implies that the coordinates  $x_i$  of the  $i$ -th particle must satisfy  $\frac{\partial L}{\partial x_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i}$ , and analogous equations hold for  $y_i$  and  $z_i$ . Altogether we thus obtain  $3N$  equations for  $N$  particles described by 3 coordinates each. In the context of Lagrangian mechanics the Euler-Lagrange equations are usually called Lagrange’s equations.

**Note:** When comparing to the section 1, we have to identify  $K = S$ ,  $x = t$  and  $(y_1(x), \dots, y_n(x)) = (\mathbf{r}_1(t), \dots, \mathbf{r}_N(t))$ .

**Proof:** The principle of least action is equivalent to Newton’s second law. If all forces are conservative this can be shown in a surprisingly simple way. The

derivative  $\frac{\partial L}{\partial \mathbf{r}_i}$  in the Lagrange equation can be written as

$$\frac{\partial L}{\partial \mathbf{r}_i} = -\frac{\partial U}{\partial \mathbf{r}_i} = \mathbf{F}_i$$

where we used that the Lagrangian depends on the particle coordinates  $\mathbf{r}_i$  only through the potential and that derivatives of the potential yield forces. On the other hand, the Lagrangian depends on the velocities  $\dot{\mathbf{r}}_i$  only through the kinetic energy. We thus obtain

$$\begin{aligned} \frac{\partial L}{\partial \dot{\mathbf{r}}_i} &= \frac{\partial T}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_i \\ \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}_i} &= m_i \ddot{\mathbf{r}}_i. \end{aligned}$$

By inserting these result into Eq. (2.1) we see that the Lagrange equations are equivalent to

$$\mathbf{F}_i = m_i \ddot{\mathbf{r}}_i,$$

i.e., Newton's second law.

### Examples:

- If there is no potential and just a single particle we have  $L = T = \frac{1}{2}m\dot{\mathbf{r}}^2$  and  $S = \int_{t(1)}^{t(2)} \frac{1}{2}m\dot{\mathbf{r}}^2(t)dt$  and the Lagrange equation reads

$$\frac{\partial L}{\partial \mathbf{r}} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}} \Rightarrow 0 = \frac{d}{dt} m\dot{\mathbf{r}} \Rightarrow \dot{\mathbf{r}} = \text{const},$$

i.e., the particle is moving on a straight line with constant velocity. (Note that if we would just demand that the length becomes stationary, we would get slightly less: we would only see that there is a straight line, but not that this line is traversed with constant velocity.)

- A particle at  $\mathbf{r} = (x, y, z)$  in a uniform gravity field has the Lagrangian

$$L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

The Lagrange equations for  $x$ ,  $y$  and  $z$  read

$$\begin{aligned} \frac{\partial L}{\partial x} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \Rightarrow 0 = \frac{d}{dt} m\dot{x} \Rightarrow \dot{x} = \text{const} \\ \frac{\partial L}{\partial y} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} \Rightarrow 0 = \frac{d}{dt} m\dot{y} \Rightarrow \dot{y} = \text{const} \\ \frac{\partial L}{\partial z} &= \frac{d}{dt} \frac{\partial L}{\partial \dot{z}} \Rightarrow -mg = \frac{d}{dt} m\dot{z} \Rightarrow \ddot{z} = -g \end{aligned}$$

As expected, we get an acceleration  $g$  in negative  $z$ -direction.

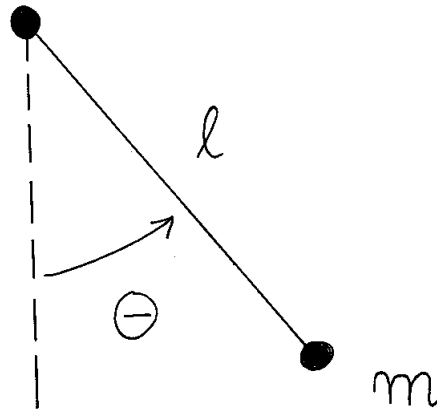


Figure 2.1: Two-dimensional pendulum.

### 2.3 Generalised coordinates and constraints

The great advantage of Lagrangian mechanics is that it can be elegantly generalised to arbitrary coordinate systems, and to systems where particles are not allowed to go into certain directions.

Let us thus consider a system of several particles, whose positions are given in arbitrary coordinates (Cartesian, polar, spherical coordinates, differences between particle positions, whatever ...). These **generalised coordinates** are then denoted by

$$q_1, q_2, \dots, q_d .$$

If the particles are **allowed to go anywhere**, the number of coordinates needed to describe each particle is given by the number of dimensions of the system (i.e. usually three). Altogether the number of variables is thus

$$d = \#\text{particles} \cdot \#\text{dimensions} .$$

However there are also situations where particles are not allowed to go in certain directions (i.e., there are **constraints** on the position of particles). In this case, we have to drop the coordinates corresponding to these directions and the number of variables is given by

$$d = \#\text{particles} \cdot \#\text{dimensions} - \#\text{constraints} . \quad (2.2)$$

**Example:** Consider a **pendulum** (see Fig. 2.1) where a mass  $m$  (the “bob”) is attached to some point through a rod of fixed length. If we take the latter point as the origin, we can use polar coordinates. But only the angular coordinate  $\theta$  changes as the pendulum swings back and forth<sup>1</sup>, the distance  $\rho$  from the origin stays fixed and can therefore be dropped as a variable. We thus need only one variable instead of two.

---

<sup>1</sup>In contrast to usual polar coordinates, this coordinate is often taken as the angle enclosed with the negative  $y$ -direction, not the positive  $x$ -direction

We will see several more complicated examples of such constraints later in the lecture. The space of all allowed positions of particles parametrised by the coordinates  $q_1, q_2, \dots, q_d$  is called the **configuration space** of the system. These coordinates are also called the **degrees of freedom** of the system.

### Relation between generalised and Cartesian coordinates

The generalised coordinates determine the Cartesian particle positions  $\mathbf{r}_i$ , i.e., we can write  $\mathbf{r}_i$  as a function of  $q_1, \dots, q_d$ . Being slightly more general, we could allow for  $\mathbf{r}_i$  to depend on times as well, and write

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_d, t) . \quad (2.3)$$

(This includes the case that e.g. the origin of our generalised coordinate system moves in time – a situation that won't occur often in this course.) Given Eq. (2.3), the particle velocities  $\dot{\mathbf{r}}_i$  can be determined using the chain rule,

$$\dot{\mathbf{r}}_i = \frac{d\mathbf{r}_i}{dt} = \sum_{\alpha=1}^d \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \mathbf{r}_i}{\partial t} . \quad (2.4)$$

Here the right-hand side involves generalised coordinates, their derivatives and time. Hence we can express  $\dot{\mathbf{r}}_i$  as a function of  $q_1, \dots, q_d, \dot{q}_1, \dots, \dot{q}_d$  and  $t$ .

### Lagrangian mechanics

We can now use these relations to express all quantities relevant for Lagrangian mechanics in terms of the new coordinates. Using (2.3) we can write the potential energy as a function

$$U = U(q_1, \dots, q_d, t) .$$

Expressing the velocities through (2.4) we can write the kinetic energy as a function

$$T = T(q_1, \dots, q_d, \dot{q}_1, \dots, \dot{q}_d, t) .$$

The Lagrangian thus turns into a function

$$L(q_1, \dots, q_d, \dot{q}_1, \dots, \dot{q}_d, t) = T(q_1, \dots, q_d, \dot{q}_1, \dots, \dot{q}_d, t) - U(q_1, \dots, q_d, t) ,$$

and the action can be written as a functional depending on the functions  $q_1(t), \dots, q_d(t)$ ,

$$S[q_1, \dots, q_d] = \int_{t(1)}^{(2)} L(q_1(t), \dots, q_d(t), \dot{q}_1(t), \dots, \dot{q}_d(t), t) dt .$$

The boundary conditions can be expressed as

$$q_\alpha(t^{(1)}) = q_\alpha^{(1)}, \quad q_\alpha(t^{(2)}) = q_\alpha^{(2)} .$$

We now claim that a result analogous to Eq. (2.1) also holds for  $q_1, \dots, q_d$ . This means that (i) the Lagrangian formulation of mechanics holds is valid for arbitrary coordinate systems and (ii) that it also holds for systems with constraints. The second point applies even though the forces that give rise to these constraints (e.g. the tension force preventing us from increasing the length of a pendulum) are typically non-conservative!

### Principle of “least” action (general form)

Consider systems where all forces are either **conservative** or give rise to **constraints**. For such systems all  $q_\alpha(t)$  are chosen such that the action  $S$  becomes stationary w.r.t. variations of  $q_\alpha(t)$  that preserve the boundary conditions at  $t^{(1)}$  and  $t^{(2)}$ . We thus have

$$\frac{\partial L}{\partial q_\alpha} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha}$$

for all  $\alpha = 1 \dots d$ .

The **proof** of this statement will be given later, after discussing some examples. (The tricky bit will be the generalisation to systems with constraints. If not for the constraints, we could give a very short proof. Essentially, we could invoke the proof for Cartesian coordinates and then argue that an extremum of the action remains an extremum of the action regardless of which system of coordinates we are working in.)

#### 2.3.1 Gravitational field

As a first example for Lagrangian mechanics with generalised coordinates let us consider the trajectory  $\mathbf{r}(t)$  of a mass  $m$  (e.g. the earth) in the gravitational field of a mass  $M$  (e.g. the sun) at the origin. The corresponding gravitational potential reads  $U = -\frac{GmM}{|\mathbf{r}|}$  and the kinetic energy is, of course, given by  $T = \frac{1}{2}m\dot{\mathbf{r}}^2$ .

The symmetry of the problem now suggests to work in **spherical coordinates**, where the vectors  $\mathbf{r}$  are parametrised by their distance from the origin  $\rho$  and two angles  $\theta$  and  $\phi$ :

$$\mathbf{r} = \rho \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}$$

In spherical coordinates the **potential energy**  $U$  turns into

$$U = -\frac{GmM}{\rho}.$$

To get the kinetic energy, we use that

$$\dot{\mathbf{r}} = \dot{\rho} \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix} + \rho \dot{\theta} \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix} + \rho \dot{\phi} \sin \theta \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}$$

where the three vectors multiplied with  $\dot{\rho}$ ,  $\rho \dot{\theta}$  and  $\rho \dot{\phi} \sin \theta$  are all normalised and perpendicular to each other. We thus get

$$T = \frac{m}{2} \dot{\mathbf{r}}^2 = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\theta}^2 + \rho^2 \dot{\phi}^2 \sin^2 \theta)$$

Note that in contrast to the Cartesian case  $T$  does not only depend on the derivatives  $\dot{\rho}$ ,  $\dot{\theta}$ ,  $\dot{\phi}$  but also on  $\rho$  and  $\theta$ . The **Lagrangian** can now be written as

$$L(\rho, \theta, \phi, \dot{\rho}, \dot{\theta}, \dot{\phi}) = T - U = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\theta}^2 + \rho^2 \dot{\phi}^2 \sin^2 \theta) + \frac{GmM}{\rho}$$

and the **action**  $S$  can be written as a functional of  $\rho(t)$ ,  $\theta(t)$ , and  $\phi(t)$ :

$$S = \int_{t^{(1)}}^{t^{(2)}} L(\rho(t), \theta(t), \phi(t), \dot{\rho}(t), \dot{\theta}(t), \dot{\phi}(t)) dt . \quad (2.5)$$

The principle of stationary action now gives rise to **Lagrange equations** for  $\rho$ ,  $\theta$  and  $\phi$ :

$$\begin{aligned} \frac{\partial L}{\partial \rho} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\rho}} &\Rightarrow m\rho\dot{\theta}^2 + m\rho\sin^2\theta\dot{\phi}^2 - \frac{GmM}{\rho^2} = m\ddot{\rho} \\ \frac{\partial L}{\partial \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &\Rightarrow m\rho^2\sin\theta\cos\theta\dot{\phi}^2 = \frac{d}{dt}(m\rho^2\dot{\theta}) \\ \frac{\partial L}{\partial \phi} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} &\Rightarrow 0 = \frac{d}{dt}(m\rho^2\dot{\phi}\sin^2\theta) \end{aligned}$$

We thus see that Lagrange's formulation of mechanics makes it simple to switch between coordinate systems: One only has to rewrite the Lagrangian in terms of the new coordinates and invoke Lagrange's equations which have the same form in every coordinate system.

### 2.3.2 Pendulum

To illustrate Lagrangian mechanics for systems with constraints, let us consider the example of the pendulum. The generalised coordinate  $-\pi < \theta < \pi$  depicted in Fig. 2.1 determines the  $x$ - and  $y$ -coordinates of the mass  $m$  as

$$\begin{aligned} x &= l \sin \theta \\ y &= -l \cos \theta . \end{aligned}$$

The derivatives of these coordinates read

$$\begin{aligned} \dot{x} &= l \cos \theta \dot{\theta} \\ \dot{y} &= l \sin \theta \dot{\theta} \end{aligned}$$

and determine the **kinetic energy** as

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\theta}^2$$

The **potential energy** is simply given by

$$U = mgy = -mgl \cos \theta$$

The **Lagrangian** thus takes the form

$$L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl \cos \theta .$$

The **Lagrange equation** for our generalised coordinate  $\theta$  now reads

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} .$$

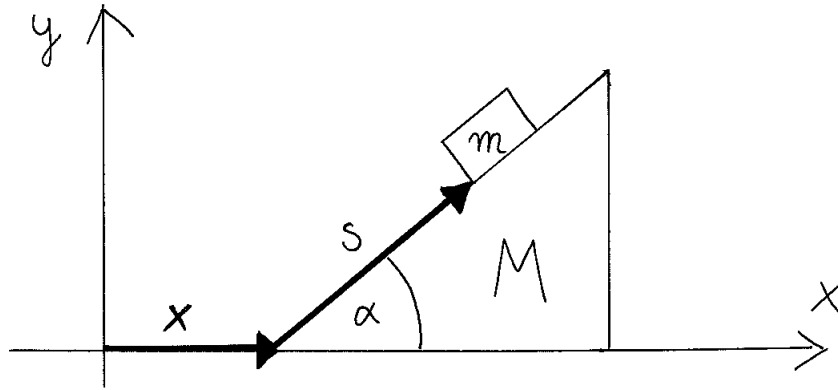


Figure 2.2: Inclined plane.

If we do the derivatives

$$\begin{aligned}\frac{\partial L}{\partial \dot{\theta}} &= ml^2 \dot{\theta} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= ml^2 \ddot{\theta} \\ \frac{\partial L}{\partial \theta} &= -mgl \sin \theta ,\end{aligned}$$

this turns into

$$\begin{aligned}-mgl \sin \theta &= ml^2 \ddot{\theta} \\ \Rightarrow \ddot{\theta} &= -\frac{g}{l} \sin \theta .\end{aligned}\tag{2.6}$$

Now one can proceed as in Newtonian mechanics, i.e, approximate  $\sin \theta$  by  $\theta$ . The resulting equation

$$\ddot{\theta} = -\frac{g}{l} \theta$$

has the solution

$$\theta(t) = A \cos \left( \sqrt{\frac{g}{l}} t + \phi \right)$$

where  $A$  and  $\phi$  are constants.

### 2.3.3 Inclined plane

As a further example, we consider an inclined plane (see Fig. 2.2). This plane slides freely on a horizontal table, and a block slides freely on the plane. The mass of the plane is  $M$ , and the mass of the block is  $m$ . We thus have two **constraints**: The block must remain on the plane and the plane must remain on the table.<sup>2</sup> Convenient **generalised coordinates** are the  $x$ -coordinate of the plane and the distance  $s$  of the block from the beginning of the plane, as sketched in Fig. 2.2. These coordinates determine the position of the plane as

$$\begin{pmatrix} x \\ 0 \end{pmatrix}$$

<sup>2</sup>Note that we are not interested in the motion of the block once it has slid down the plane.



and the position of the block as

$$\mathbf{r} = \begin{pmatrix} x + s \cos \alpha \\ s \sin \alpha \end{pmatrix}$$

The **kinetic energy** of the plane reads  $\frac{1}{2}M\dot{x}^2$ , whereas the block has the kinetic energy  $\frac{1}{2}m\dot{\mathbf{r}}^2$ . To express  $\frac{1}{2}m\dot{\mathbf{r}}^2$  in terms of our generalised coordinates we write

$$\begin{aligned} \dot{\mathbf{r}} &= \begin{pmatrix} \dot{x} + \dot{s} \cos \alpha \\ \dot{s} \sin \alpha \end{pmatrix} \\ \dot{\mathbf{r}}^2 &= (\dot{x} + \dot{s} \cos \alpha)^2 + (\dot{s} \sin \alpha)^2 = \dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s} \cos \alpha . \end{aligned}$$

The overall kinetic energy is thus obtained as

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{s}^2 + 2\dot{x}\dot{s} \cos \alpha) = \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s} \cos \alpha .$$

The only contribution to the **potential energy**  $U$  is due to the height of the block,

$$U = mgs \sin \alpha .$$

The **Lagrangian** thus reads

$$L = T - U = \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s} \cos \alpha - mgs \sin \alpha .$$

We now obtain two **Lagrange equations**, one for the coordinate  $x$  and one for  $s$ . For  $x$  we get

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 .$$

With the derivatives

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= (M + m)\dot{x} + m \cos \alpha \dot{s} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} &= (M + m)\ddot{x} + m \cos \alpha \ddot{s} \\ \frac{\partial L}{\partial x} &= 0 \end{aligned}$$

this turns into

$$-(M + m)\ddot{x} - m \cos \alpha \ddot{s} = 0 . \quad (2.7)$$

The Lagrange equation for  $s$  reads

$$\frac{\partial L}{\partial s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{s}} = 0 .$$

If we use the derivatives

$$\begin{aligned} \frac{\partial L}{\partial \dot{s}} &= m\dot{s} + m \cos \alpha \dot{x} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{s}} &= m\ddot{s} + m \cos \alpha \ddot{x} \\ \frac{\partial L}{\partial s} &= -mg \sin \alpha \end{aligned}$$

and cancel the factors  $-m$  this simplifies to

$$\ddot{s} + \cos \alpha \ddot{x} + g \sin \alpha = 0. \quad (2.8)$$

We have thus obtained two coupled equations (2.7), (2.8) for the second derivatives  $\ddot{x}$  and  $\ddot{s}$ . To obtain separate equations for  $\ddot{x}$  and  $\ddot{s}$ , we solve (2.7) for  $\ddot{x}$  and use it to eliminate  $\ddot{x}$  in (2.8). This yields

$$\begin{aligned} \ddot{s} - \frac{m}{M+m} \cos^2 \alpha \ddot{s} + g \sin \alpha &= 0 \\ \Rightarrow (M+m)\ddot{s} - m \cos^2 \alpha \ddot{s} + (M+m)g \sin \alpha &= 0 \\ \Rightarrow (M+m \sin^2 \alpha)\ddot{s} + (M+m)g \sin \alpha &= 0 \end{aligned}$$

which finally leads to

$$\ddot{s} = -\frac{(M+m)g \sin \alpha}{M+m \sin^2 \alpha} = \text{const}. \quad (2.9)$$

If we substitute (2.9) back into (2.7) we get

$$\ddot{x} = \frac{mg}{M+m \sin^2 \alpha} \sin \alpha \cos \alpha = \text{const}. \quad (2.10)$$

Eq. (2.9) and (2.10) give the second derivatives of our generalised coordinates as constants depending on the parameters of the problem. One easily checks that in special cases like  $\alpha \rightarrow 0$ ,  $\alpha \rightarrow \frac{\pi}{2}$  or  $m \rightarrow 0$  these results agree with what we might expect. (E.g. for a perpendicular plane with  $\alpha = \frac{\pi}{2}$  the block simply falls down with  $\ddot{s} = -g$  whereas the plane stays fixed.) Assuming that the block and the plane are initially at rest we can integrate (2.9) and (2.10), to get

$$\begin{aligned} x(t) &= x(0) + \frac{1}{2} \ddot{x} t^2 \\ s(t) &= s(0) + \frac{1}{2} \ddot{s} t^2. \end{aligned}$$

### For comparison: Inclined plane with Newton

It is instructive to compare the above treatment of the inclined plane with Newtonian mechanics. The main difference will be that in Newtonian mechanics the constraints can no longer be built in by choosing appropriate generalised coordinates. Instead one has to work in Cartesian coordinates and take into account **all** forces acting on the plane and the block; in particular this includes forces that make sure that the constraints are satisfied.

### Forces acting on the inclined plane

For the inclined plane Newton's law implies

$$\mathbf{F}_P = M \mathbf{a}_P$$

where

$$\mathbf{a}_P = \begin{pmatrix} \ddot{x} \\ 0 \end{pmatrix}$$

is the acceleration of the plane and  $\mathbf{F}_P$  is the sum of all forces acting on the plane (see Fig. 2.3):

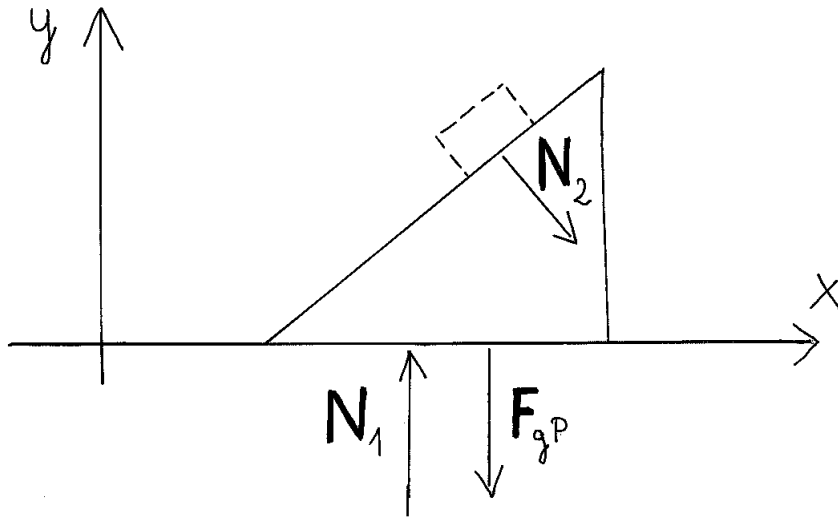


Figure 2.3: Forces acting on the inclined plane.

- This includes the gravitational force

$$\mathbf{F}_{gP} = \begin{pmatrix} 0 \\ -Mg \end{pmatrix}.$$

- Moreover we have the constraint that the plane stays on the table. Hence there must be a force coming from the table which keeps the plane from “falling through” the table. This force is due to the rigidity of the table (and can be felt by pressing on a table!) It must be oriented in a vertical direction, i.e., normal to the table, and thus be of the form

$$\mathbf{N}_1 = \begin{pmatrix} 0 \\ N_1 \end{pmatrix}$$

where  $N_1$  is an undetermined constant.

- In addition the block exerts a force on the plane. Due to the geometry of the system, this force should press the plane downwards, but also push it to the right. The precise direction of this force should be normal to the upper side of the inclined plane. This is easily understood if we realise that the force could never cause a tangential motion of the plane along the interface between the block and the plane. The normal force should thus be of the form (compare Fig. 2.3)

$$\mathbf{N}_2 = \begin{pmatrix} N_2 \sin \alpha \\ -N_2 \cos \alpha \end{pmatrix}.$$

The overall force acting on the plane is therefore given by

$$\mathbf{F}_P = \mathbf{F}_{gP} + \mathbf{N}_1 + \mathbf{N}_2 = \begin{pmatrix} N_2 \sin \alpha \\ -Mg + N_1 - N_2 \cos \alpha \end{pmatrix}$$

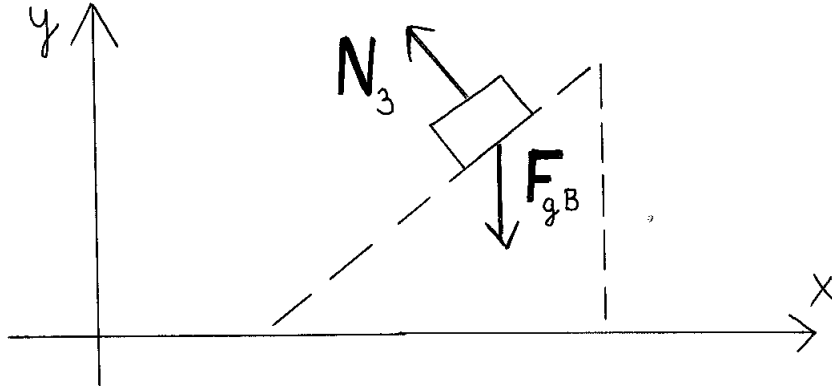


Figure 2.4: Forces acting on the block.

where  $N_1$  and  $N_2$  are undetermined constants. Now the two components of the vector equation  $\mathbf{F}_P = M\mathbf{a}_P$  read

$$N_2 \sin \alpha = M\ddot{x} \Rightarrow N_2 = \frac{M\ddot{x}}{\sin \alpha} \quad (2.11)$$

and

$$-Mg + N_1 - N_2 \cos \alpha = 0 \Rightarrow N_1 = Mg + N_2 \cos \alpha . \quad (2.12)$$

Inserting the former into the latter equation we obtain

$$N_1 = M(g + \ddot{x} \cot \alpha) .$$

### Forces acting on the block

For the block Newton's law assumes the form

$$\mathbf{F}_B = m\mathbf{a}_B$$

with the acceleration

$$\mathbf{a}_B = \frac{d^2 \mathbf{r}}{dt^2} = \frac{d^2}{dt^2} \begin{pmatrix} x + s \cos \alpha \\ s \sin \alpha \end{pmatrix} = \begin{pmatrix} \ddot{x} + \ddot{s} \cos \alpha \\ \ddot{s} \sin \alpha \end{pmatrix} .$$

The force  $\mathbf{F}_B$  is a sum of

- the gravitational force

$$\mathbf{F}_{gB} = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$$

- and a force  $\mathbf{N}_3$  exerted by the plane on the block. The latter originates from the rigidity of the plane and makes sure that the block does not fall through the plane. Hence it is another force of constraint. Due to Newton's third law it must be equal to the negative of the force that the block exerts on the plane. We thus have (using (2.11))

$$\mathbf{N}_3 = -\mathbf{N}_2 = \begin{pmatrix} -N_2 \sin \alpha \\ N_2 \cos \alpha \end{pmatrix} = \frac{M\ddot{x}}{\sin \alpha} \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} .$$

Summation yields the overall force

$$\mathbf{F}_B = \mathbf{F}_{gB} + \mathbf{N}_3 = \begin{pmatrix} -M\ddot{x} \\ -mg + M\ddot{x} \cot \alpha \end{pmatrix}.$$

Now the two components of Newton's law read

$$\begin{aligned} m(\ddot{x} + \ddot{s} \cos \alpha) &= -M\ddot{x} \\ \Rightarrow \ddot{x} &= -\frac{m}{m+M} \ddot{s} \cos \alpha \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} m\ddot{s} \sin \alpha &= -mg + M\ddot{x} \cot \alpha \\ &= -mg - \frac{mM}{m+M} \ddot{s} \frac{\cos^2 \alpha}{\sin \alpha} \end{aligned} \quad (2.14)$$

(where in the last step we used Eq. (2.13)). If we multiply Eq. (2.14) with  $\sin \alpha \frac{m+M}{m}$  we obtain

$$\begin{aligned} (m+M)\ddot{s} \sin^2 \alpha &= -(m+M)g \sin \alpha - M\ddot{s} \cos^2 \alpha \\ \Rightarrow (m \sin^2 \alpha + M)\ddot{s} &= -(m+M)g \sin \alpha \\ \Rightarrow \ddot{s} &= -\frac{(M+m)g \sin \alpha}{M+m \sin^2 \alpha} = \text{const.} \end{aligned} \quad (2.15)$$

Substitution into (2.13) then yields

$$\ddot{x} = \frac{mg}{M+m \sin^2 \alpha} \sin \alpha \cos \alpha = \text{const.} \quad (2.16)$$

## Discussion

We see that Lagrange's equations and Newton's law yield coinciding results, given in Eqs. (2.9), (2.10) and in Eqs. (2.15), (2.16). The Lagrangian treatment is considerably simpler since we can build in the constraints by appropriately choosing coordinates. In the Newtonian approach we instead have to take into account additional forces. These forces  $\mathbf{N}_1$ ,  $\mathbf{N}_2$ ,  $\mathbf{N}_3$  are forces of constraint. They keep the block on the plane and the plane on the table. These forces don't appear in the Lagrangian treatment. However we can always determine them if we want (e.g., if we need the force exerted on the table to see if it breaks). We then have to solve Lagrange's equations and determine the forces of constraint from  $m\ddot{\mathbf{r}}_i$  and the known conservative forces  $\mathbf{F}^{\text{pot}}$  as in

$$\begin{aligned} m_i \ddot{\mathbf{r}}_i &= \mathbf{F}_i^{\text{pot}} + \mathbf{F}_i^{\text{constraint}} \\ \Rightarrow \mathbf{F}_i^{\text{constraint}} &= \mathbf{F}_i^{\text{pot}} - m_i \ddot{\mathbf{r}}_i. \end{aligned}$$

### 2.3.4 General properties of forces of constraint

We will now discuss forces of constraint in more general terms. We want to speak of a **force of constraint** when a force makes sure that the constraints are satisfied but has no further impact on the motion. Such forces **must point in directions where the particles are forbidden from going** and compensate all other forces that may point in these directions. They may have no components in directions

where the particles are allowed to go. An example is the normal force  $\mathbf{N}_1$  in the example of the inclined plane, see Fig. 2.5. This force points in the direction normal to the table, and compensates the gravitational force and the force from the block which push the inclined plane into a forbidden direction.

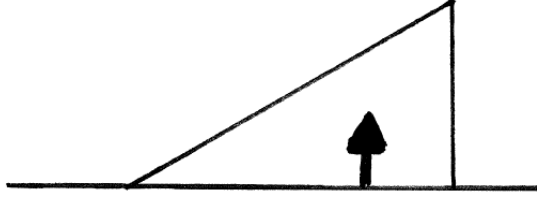


Figure 2.5: Example for a force of constraint.

### Allowed and forbidden directions

To formulate the above condition mathematically we have to look in more detail at the directions in which a system is, or is not, allowed to move.

- First of all we note that all particle positions  $(\mathbf{r}_1, \dots, \mathbf{r}_N)$  form a  $3N$ -dimensional space (or a  $2N$ -dimensional space if the position vectors are in  $\mathbb{R}^2$ ).
- However not all positions are allowed. We parametrize the allowed positions of particles by  $d$  generalised coordinates  $q_1, \dots, q_d$  and (possibly) time as

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_d, t).$$

This defines the  $d$ -dimensional configuration space.

- Starting from an allowed  $(\mathbf{r}_1, \dots, \mathbf{r}_N)$  we can go into all directions in the  $3N$ -dimensional space that can be reached by changing the  $q_1, \dots, q_d$ . For instance, if we change  $q_\alpha$  by an infinitesimal amount  $\delta q_\alpha$ , each particle position  $\mathbf{r}_i$  changes by  $\delta \mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \delta q_\alpha$ . This means that all infinitesimal motions in the  $3N$ -dimensional space that go in directions

$$\left( \frac{\partial \mathbf{r}_1}{\partial q_\alpha}, \dots, \frac{\partial \mathbf{r}_N}{\partial q_\alpha} \right) \quad (2.17)$$

are allowed. The same applies to linear combinations of these directions. At every point  $(\mathbf{r}_1, \dots, \mathbf{r}_N)$  there are  $d$  directions as in (2.17), one for each  $\alpha = 1, \dots, d$ .

- This leaves  $3N - d$  linearly independent forbidden directions (one for each constraint). They are perpendicular to the allowed directions.

Now consider a set of forces  $(\mathbf{F}_1^c, \mathbf{F}_2^c, \dots, \mathbf{F}_N^c)$ , where  $\mathbf{F}_1^c$  acts on the first particle,  $\mathbf{F}_2^c$  acts on the second particle, etc. These forces are forces of constraint if the  $3N$ -dimensional vector  $(\mathbf{F}_1^c, \mathbf{F}_2^c, \dots, \mathbf{F}_N^c)$  points into a forbidden direction (such that other forces pointing into these directions are compensated) and is perpendicular to

the allowed directions (such that the motion in allowed directions is not affected). This means that the scalar product of  $(\mathbf{F}_1^c, \mathbf{F}_2^c, \dots, \mathbf{F}_N^c)$  and any of the allowed directions must vanish. Forces of constraint can thus be defined as follows:

**Definition (D'Alembert's principle)**

A set of forces  $(\mathbf{F}_1^c, \mathbf{F}_2^c, \dots, \mathbf{F}_N^c)$  are **forces of constraint** if we have

$$(\mathbf{F}_1^c, \mathbf{F}_2^c, \dots, \mathbf{F}_N^c) \cdot \left( \frac{\partial \mathbf{r}_1}{\partial q_\alpha}, \dots, \frac{\partial \mathbf{r}_N}{\partial q_\alpha} \right) = \sum_{i=1}^N \mathbf{F}_i^c \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = 0 \quad (2.18)$$

for all  $\alpha = 1 \dots d$

**Note:** The term “allowed direction” used here is not standard. Traditionally, these allowed directions are rather referred to a “**virtual displacements**”. Furthermore note that the left-hand side of Eq. (2.18) has the dimension of work, since it involves a product of forces and changes of positions. The traditional way of stating D'Alembert's principle is thus to say that virtual displacements **do no work**.

### 2.3.5 Derivation of Lagrange's equations from Newton's law in the general case

We have now learnt enough about constraints to give a proof for Lagrange's equations and the principle of least action for **generalised coordinates** and **systems with constraints**. We will see that d'Alembert's principle is crucial for showing that forces of constraint don't spoil the picture.

We consider systems where all forces are either conservative forces or forces of constraint. For these systems **Newton's** second law reads:

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i = -\frac{\partial U}{\partial \mathbf{r}_i} + \mathbf{F}_i^c \quad (2.19)$$

where  $\mathbf{F}_i$  is the overall force acting on the  $i$ -th particle. It is written as a sum of the conservative force  $-\frac{\partial U}{\partial \mathbf{r}_i}$  and the force of constraint  $\mathbf{F}_i^c$ . We now want to show that **Lagrange's** equations hold as well. I.e., if we parametrise the particle positions by generalised coordinates where the constraint are automatically built in,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_d, t), \quad (2.20)$$

then we have

$$\frac{\partial L}{\partial q_\alpha} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} = 0 \quad (2.21)$$

for all  $\alpha = 1 \dots d$ . This means that Hamilton's principle holds, i.e., that particles travel on trajectories for which the action  $S = \int L dt$  becomes extremal.

#### Preparation: Formulas for partial derivatives of $\dot{\mathbf{r}}_i$

To prepare for a proof, we first derive formulas for the derivatives of the velocities  $\dot{\mathbf{r}}_i$  w.r.t. the generalised coordinates  $q_\alpha$  and their derivatives  $\dot{q}_\alpha$ . In Eq. (2.4) we

took the derivative of

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_d, t), \quad (2.22)$$

using the chain rule, and got

$$\dot{\mathbf{r}}_i = \frac{d\mathbf{r}_i}{dt} = \sum_{\beta=1}^d \frac{\partial \mathbf{r}_i}{\partial q_\beta}(q_1, \dots, q_d, t) \dot{q}_\beta + \frac{\partial \mathbf{r}_i}{\partial t}(q_1, \dots, q_d, t). \quad (2.23)$$

We thus expressed the velocity as a function

$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i(q_1, \dots, q_d, \dot{q}_1, \dots, \dot{q}_d, t).$$

We can now take derivatives of  $\dot{\mathbf{r}}_i$  w.r.t. generalised coordinates and  $\dot{q}_\beta$ 's. If we look at (2.23) we see that the derivative w.r.t.  $\dot{q}_\beta$  is just the term multiplying  $\dot{q}_\beta$ . We thus have

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_\beta} = \frac{\partial \mathbf{r}_i}{\partial q_\beta}. \quad (2.24)$$

Since the two dots on the left-hand side have disappeared on the right-hand side, this rule is also known as the “**cancellation of dots**”. For the derivatives w.r.t. generalised coordinates we will show that

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial q_\alpha} = \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_\alpha}. \quad (2.25)$$

This means that the total derivative w.r.t.  $t$  (which appears on the left-hand side as a dot) and the partial derivative w.r.t.  $q_\alpha$  can be interchanged – which would be trivial if both derivatives were partial, but it is necessary to give a proof because one of the derivatives is a total one.

**Proof:** To evaluate the l.h.s., we use (2.23). The only  $q_\alpha$ -dependent terms in (2.23) are  $\frac{\partial \mathbf{r}_i}{\partial q_\beta}$  and  $\frac{\partial \mathbf{r}_i}{\partial t}$ . Hence the chain rule yields

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial q_\alpha} = \sum_{\beta=1}^d \frac{\partial^2 \mathbf{r}_i}{\partial q_\alpha \partial q_\beta} \dot{q}_\beta + \frac{\partial^2 \mathbf{r}_i}{\partial q_\alpha \partial t}.$$

To compute the term  $\frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_\alpha}$  on the r.h.s., we use  $\frac{\partial \mathbf{r}_i}{\partial q_\alpha}$  is a function of the generalised coordinates and time. Hence the chain rule brings about partial derivatives with respect to these quantities, and the final result

$$\frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = \sum_{\beta=1}^d \frac{\partial^2 \mathbf{r}_i}{\partial q_\alpha \partial q_\beta} \dot{q}_\beta + \frac{\partial^2 \mathbf{r}_i}{\partial q_\alpha \partial t}$$

agrees with the left-hand side. Eq. (2.25) is thus proven.

### Strategy

We are now ready to prove Lagrange's equation. We start from Newton's law, multiply both sides with  $\frac{\partial \mathbf{r}_i}{\partial q_\alpha}$ , and sum over  $i$ . This yields

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = \sum_{i=1}^N \underbrace{\left( -\frac{\partial U}{\partial \mathbf{r}_i} + \mathbf{F}_i^c \right)}_{=\mathbf{F}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}. \quad (2.26)$$



A motivation for this strategy is that we would like to get an expression that involves partial derivatives w.r.t.  $q_\alpha$  – hence it is a good idea to multiply with  $\frac{\partial \mathbf{r}_i}{\partial q_\alpha}$ . We now have to simplify all three terms obtained to get Lagrange.

### Acceleration term

For the first term involving the acceleration  $\ddot{\mathbf{r}}_i$ , we pull one of the two time derivatives in front such that it also acts on  $\frac{\partial \mathbf{r}_i}{\partial q_\alpha}$ , and then subtract the term where the time derivative acts on  $\frac{\partial \mathbf{r}_i}{\partial q_\alpha}$ ,

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = \sum_{i=1}^N m_i \left[ \frac{d}{dt} \left( \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \right].$$

We then use Eq. (2.24) for the first term, and Eq. (2.25) for the second term,

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = \sum_{i=1}^N m_i \left[ \frac{d}{dt} \left( \underbrace{\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_\alpha}}_{=\frac{1}{2} \frac{\partial}{\partial \dot{q}_\alpha} \dot{\mathbf{r}}_i^2} \right) - \underbrace{\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_\alpha}}_{=\frac{1}{2} \frac{\partial}{\partial q_\alpha} \dot{\mathbf{r}}_i^2} \right] \quad (2.27)$$

As indicated in Eq. (2.27), the terms  $\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_\alpha}$  and  $\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_\alpha}$  thus obtained can be written as derivatives of  $\dot{\mathbf{r}}_i^2$ . This is reassuring since  $\dot{\mathbf{r}}_i^2$  shows up in the kinetic energy  $T$  included in  $L$ . We may thus hope to obtain derivatives of  $T$ . To do so we now pull all derivatives in front which leads to

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\alpha} \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 - \frac{\partial}{\partial q_\alpha} \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2.$$

We now recognise  $\sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$  as the kinetic energy and write

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha}.$$

We have thus expressed the first term in (2.26) through derivatives of the kinetic energy, and obtained the intermediate result

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} = \sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}. \quad (2.28)$$

In Eq. (2.28) we have not yet used our assumptions on the forces (i.e. that they are sums of conservative forces and forces of constraint). Eq. (2.28) can thus be seen as a **generalisation of Lagrange's equation for arbitrary forces**.

### Potential term

The derivative of the potential in Eq. (2.26), multiplied with  $\frac{\partial \mathbf{r}_i}{\partial q_\alpha}$  and summed over, simply gives

$$-\sum_{i=1}^N \frac{\partial U}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = -\frac{\partial U}{\partial q_\alpha}$$

– again a term that we would like to see in Lagrange's equation!

**Constraint term**

The final term in (2.26) reads

$$\sum_{i=1}^N \mathbf{F}_i^c \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}$$

This is exactly the scalar product of forces of constraint and allowed directions (2.18) that vanishes due to d'Alembert's principle. We thus have

$$\sum_{i=1}^N \mathbf{F}_i^c \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = 0,$$

and we see that due to d'Alembert's principle the forces of constraint drop from our equations of motion.

**Result**

Summation of all three terms yields

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} &= -\frac{\partial U}{\partial q_\alpha} \\ \Rightarrow \frac{\partial(T-U)}{\partial q_\alpha} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} &= 0 \end{aligned}$$

which is suspiciously close to Lagrange's equations for  $L = T - U$ . The only thing that one might miss is a derivative  $\frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\alpha}$ . But by definition, the potential is independent of  $\dot{q}_\alpha$  and thus  $\frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\alpha} = 0$ . If we now simply add  $\frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\alpha}$  and replace  $T - U$  by  $L$ , we obtain our desired result:

**Lagrange's equation**

$$\frac{\partial L}{\partial q_\alpha} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} = 0 \quad \text{for all } \alpha = 1, \dots, d$$

## 2.4 Conserved quantities

In mechanics it is often helpful to look for **conserved quantities** and for example check whether the energy, the momentum or the angular momentum of a particle remains fixed.

**Def.:** A quantity  $A$  is a **conserved** if the total time derivative  $\frac{dA}{dt}$  vanishes.

We will show that Lagrangian mechanics provides an ideal framework to study these conserved quantities. In particular we will see that conserved quantities always arise when the Lagrangian is independent of one of the arguments  $q_1, \dots, q_d, t$ .

### 2.4.1 Energy conservation

First of all we will show that if the Lagrangian does not depend on time, the so-called generalised energy is conserved. (Usually this is just the energy itself.)

#### Conservation of the generalised energy

If the Lagrangian of a system is **independent of the time**  $t$ , i.e.,

$$L = L(\underbrace{q_1, \dots, q_d}_{=\mathbf{q}}, \underbrace{\dot{q}_1, \dots, \dot{q}_d}_{=\dot{\mathbf{q}}})$$

then the **generalised energy**

$$h \equiv \sum_{\alpha=1}^d \frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha - L = \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L$$

is a conserved quantity.

Note that here we used the vector notation  $\mathbf{q} = (q_1, \dots, q_d)$  for the generalised coordinates.

#### Proof

- Remembering variational calculus we can use the **alternative version of the Euler-Lagrange equation**. We had seen that if a function  $f = f(\mathbf{y}, \mathbf{y}')$  is independent of  $x$ , then the functional  $K = \int_{x_1}^{x_2} f(\mathbf{y}(x), \mathbf{y}'(x)) dx$  becomes stationary if

$$\frac{\partial f}{\partial \mathbf{y}'} \cdot \mathbf{y}' - f = \text{const.}$$

(Here the sign of the constant is flipped compared to section 1.) Thus the Euler-Lagrange equation was directly formulated in terms of a conservation law. To apply this result to Lagrangian mechanics, we replace  $x \rightarrow t$ ,  $\mathbf{y} \rightarrow \mathbf{q}$ ,  $f \rightarrow L$  and  $K \rightarrow S$ . Then we immediately obtain the statement above.

- Since it was so simple, we can just redo the proof and take the total derivative

of  $h$ :

$$\begin{aligned}\frac{dh}{dt} &= \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} \right) - \frac{dL}{dt} \\ &= \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \cdot \dot{\mathbf{q}} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \ddot{\mathbf{q}} - \frac{\partial L}{\partial \mathbf{q}} \dot{\mathbf{q}} - \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \ddot{\mathbf{q}}\end{aligned}$$

If we use Lagrange's equations to replace  $\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}}$  by  $\frac{\partial L}{\partial \mathbf{q}}$  we see that all terms above cancel and thus

$$\frac{dh}{dt} = 0.$$

### Example

Let us consider the motion of a particle in one dimension. The corresponding Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - U(x)$$

does not depend explicitly on time. Hence the generalised energy must be conserved; it reads

$$\begin{aligned}h &= \frac{\partial L}{\partial \dot{x}} \dot{x} - L \\ &= m\dot{x}\dot{x} - \left( \frac{1}{2}m\dot{x}^2 - U(x) \right) \\ &= \frac{1}{2}m\dot{x}^2 + U(x)\end{aligned}\tag{2.29}$$

which is just the sum of kinetic and potential energy, i.e., the **energy**  $E = T + U$  itself.

I now want to show that this result  $h = E$  actually generalises to most practical cases (albeit there are exceptions). To check this we first need an explicit formula for the Lagrangian, and in particular for the kinetic energy of a mechanical system. We can then compute  $h$  and see whether it agrees with  $E$ .

### General formula for the kinetic energy

If we use Cartesian coordinates for the particle positions  $\mathbf{r}_i$ , the kinetic energy of a mechanical system always has the form

$$T = \sum_{i=1}^N \frac{1}{2}m_i \dot{\mathbf{r}}_i^2.\tag{2.30}$$

We now want to see how this formula looks if we use generalised coordinates  $q_1, \dots, q_d$ . The particle positions can then be parametrised by these generalised coordinates and (possibly) time as

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_d, t).$$

According to the chain rule the velocity turns into

$$\dot{\mathbf{r}}_i = \sum_{\alpha=1}^d \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \mathbf{r}_i}{\partial t}.\tag{2.31}$$

If we insert this into the above formula for  $T$  we get

$$T = \sum_{i=1}^N \frac{1}{2} m_i \left[ \sum_{\alpha=1}^d \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \dot{q}_\alpha \cdot \sum_{\beta=1}^d \frac{\partial \mathbf{r}_i}{\partial q_\beta} \dot{q}_\beta + 2 \sum_{\alpha=1}^d \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \dot{q}_\alpha \cdot \frac{\partial \mathbf{r}_i}{\partial t} + \left( \frac{\partial \mathbf{r}_i}{\partial t} \right)^2 \right]. \quad (2.32)$$

This follows directly by inserting (2.31) into (2.30); the only nontrivial point is that when squaring  $\sum_{\alpha=1}^d \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \dot{q}_\alpha$  I wrote the two factors with different summation variables  $\alpha$  and  $\beta$  to make clear that there are two different sums, not just one. It would be nice if we could write Eq. (2.32) in a less clunky way. To do so we abbreviate the terms that are not derivatives of  $q$  by

$$\begin{aligned} M_{\alpha\beta}(q_1, \dots, q_d, t) &\equiv \sum_{i=1}^N m_i \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \cdot \frac{\partial \mathbf{r}_i}{\partial q_\beta} \\ v_\alpha(q_1, \dots, q_d, t) &\equiv \sum_{i=1}^N m_i \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \cdot \frac{\partial \mathbf{r}_i}{\partial t} \\ c(q_1, \dots, q_d, t) &\equiv \sum_{i=1}^N m_i \left( \frac{\partial \mathbf{r}_i}{\partial t} \right)^2. \end{aligned} \quad (2.33)$$

The kinetic energy then turns into

$$T = \frac{1}{2} \sum_{\alpha=1}^d \sum_{\beta=1}^d M_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta + \sum_{\alpha=1}^d v_\alpha \dot{q}_\alpha + \frac{c}{2}. \quad (2.34)$$

To make this result even more compact we adopt a vector and matrix notation. We thus collect all  $M_{\alpha\beta}$ 's into a matrix

$$M \equiv \begin{pmatrix} M_{11} & \dots & M_{1d} \\ \vdots & & \vdots \\ M_{d1} & \dots & M_{dd} \end{pmatrix}$$

and all  $v_\alpha$ 's into a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}.$$

Here  $M$  is a symmetric matrix ( $M_{\alpha\beta} = M_{\beta\alpha}$ ) because the matrix elements  $M_{\alpha\beta}$  defined above remain the same if  $\alpha$  and  $\beta$  are interchanged. With this  $M$  and  $\mathbf{v}$  the kinetic energy assumes the form:

**General formula for the kinetic energy**

$$T = \frac{1}{2} \dot{\mathbf{q}} \cdot M \dot{\mathbf{q}} + \mathbf{v} \cdot \dot{\mathbf{q}} + \frac{c}{2} \quad (2.35)$$

The term quadratic in  $\dot{\mathbf{q}}$ ,  $\frac{1}{2} \dot{\mathbf{q}} \cdot M \dot{\mathbf{q}}$ , looks very much like the usual formula for the kinetic energy of a single particle in Cartesian coordinates. The only differences are that  $\dot{\mathbf{q}}$  is a vector containing derivatives of generalised coordinates, and that the

mass is replaced by a matrix that may depend on  $\mathbf{q}$ . This matrix is also called the **mass matrix**.

The linear and constant terms are new. However they show up only if our transformation between Cartesian and generalised coordinates involves time, i.e., if  $\mathbf{r}_i = (q_1, \dots, q_d, t)$  with  $\frac{\partial \mathbf{r}_i}{\partial t} \neq 0$ . In the usual situation that there is no time dependence, we have  $\frac{\partial \mathbf{r}_i}{\partial t} = 0$  and the  $v_\alpha$  and  $c$  defined in (2.33) simply vanish. Then we only have the quadratic term.

### Formulas for gradients of linear and quadratic functions

To get  $h$  we must evaluate derivatives of  $L = T - U$  w.r.t.  $\mathbf{q}$  and  $\dot{\mathbf{q}}$ , i.e., we must learn how to deal with derivatives of scalar products like  $\mathbf{v} \cdot \mathbf{q}$  and quadratic terms like  $\dot{\mathbf{q}} \cdot M \dot{\mathbf{q}}$  with respect to the vectors involved. We recall that the derivative (gradient) w.r.t. a vector  $\mathbf{u}$  is defined as the vector containing partial derivatives w.r.t. the components of  $\mathbf{u}$ , i.e.,

$$\frac{\partial f}{\partial \mathbf{u}} = \begin{pmatrix} \frac{\partial f}{\partial u_1} \\ \vdots \\ \frac{\partial f}{\partial u_d} \end{pmatrix}.$$

We will show that derivatives of such terms are given by

$$\frac{\partial(\mathbf{v} \cdot \mathbf{u})}{\partial \mathbf{u}} = \mathbf{v} \quad (2.36)$$

$$\frac{\partial(\mathbf{u} \cdot M \mathbf{u})}{\partial \mathbf{u}} = 2M\mathbf{u} \quad \text{if } M \text{ is symmetric} \quad (2.37)$$

which are just the formulas we would get if  $\mathbf{v}$ ,  $\mathbf{u}$  and  $M$  were scalars.

**Proof of (2.36):** We use that

$$\mathbf{v} \cdot \mathbf{u} = \sum_{\alpha=1}^d v_\alpha u_\alpha.$$

The partial derivatives are thus given by

$$\frac{\partial(\mathbf{v} \cdot \mathbf{u})}{\partial u_\alpha} = v_\alpha$$

and collecting them into a vector yields  $\mathbf{v}$  as in (2.36).

**Proof of (2.37):** We use that

$$\mathbf{u} \cdot M \mathbf{u} = \sum_{\alpha=1}^d \sum_{\beta=1}^d M_{\alpha\beta} u_\alpha u_\beta.$$

The partial derivatives are thus given by

$$\frac{\partial(\mathbf{u} \cdot M \mathbf{u})}{\partial u_\gamma} = \sum_{\alpha=1}^d \sum_{\beta=1}^d M_{\alpha\beta} \left( \frac{\partial u_\alpha}{\partial u_\gamma} u_\beta + u_\alpha \frac{\partial u_\beta}{\partial u_\gamma} \right). \quad (2.38)$$

Here the derivative  $\frac{\partial u_\alpha}{\partial u_\gamma}$  reads

$$\frac{\partial u_\alpha}{\partial u_\gamma} = \delta_{\alpha\gamma} \equiv \begin{cases} 1 & \text{if } \alpha = \gamma \\ 0 & \text{otherwise} \end{cases}$$

Therefore the first sum in (2.38) only receives contributions when  $\alpha = \gamma$ . We thus drop  $\frac{\partial u_\alpha}{\partial u_\gamma}$  and the summation over  $\alpha$  and replace the remaining  $\alpha$  by  $\gamma$ . The  $\frac{\partial u_\beta}{\partial u_\gamma}$  in the second term is handled in an analogous way. We then obtain

$$\frac{\partial(\mathbf{u} \cdot M\mathbf{u})}{\partial u_\gamma} = \sum_{\beta=1}^d M_{\gamma\beta} u_\beta + \sum_{\alpha=1}^d M_{\alpha\gamma} u_\alpha.$$

If we now rename the summation variable  $\alpha$  in the second sum into  $\beta$  and use that  $M$  is symmetric ( $M_{\beta\gamma} = M_{\gamma\beta}$ ) we get

$$\frac{\partial(\mathbf{u} \cdot M\mathbf{u})}{\partial u_\gamma} = \sum_{\beta=1}^d M_{\gamma\beta} u_\beta + \sum_{\beta=1}^d \underbrace{M_{\beta\gamma}}_{=M_{\gamma\beta}} u_\beta = 2 \sum_{\beta=1}^d M_{\gamma\beta} u_\beta$$

Collecting all partial derivatives  $\frac{\partial(\mathbf{u} \cdot M\mathbf{u})}{\partial u_\gamma}$  into a vector we thus get  $2M\mathbf{u}$  as claimed.

### Generalised energy

We are now prepared to evaluate the generalised energy. Due to (2.35) the Lagrangian is given by

$$L = \frac{1}{2} \dot{\mathbf{q}} \cdot M \dot{\mathbf{q}} + \mathbf{v} \cdot \dot{\mathbf{q}} + \frac{c}{2} - U.$$

The generalised energy can now be computed from (2.29),

$$\begin{aligned} h &= \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L \\ &= (M \dot{\mathbf{q}} + \mathbf{v}) \cdot \dot{\mathbf{q}} - \frac{1}{2} \dot{\mathbf{q}} \cdot M \dot{\mathbf{q}} - \mathbf{v} \cdot \dot{\mathbf{q}} - \frac{c}{2} + U \\ &= \frac{1}{2} \dot{\mathbf{q}} \cdot M \dot{\mathbf{q}} - \frac{c}{2} + U \end{aligned}$$

For comparison, the energy is

$$E = T + U = \frac{1}{2} \dot{\mathbf{q}} \cdot M \dot{\mathbf{q}} + \mathbf{v} \cdot \dot{\mathbf{q}} + \frac{c}{2} + U.$$

So the good news is:

**If the kinetic energy is quadratic in  $\dot{\mathbf{q}}$ , i.e.  $\mathbf{v} = 0$  and  $c = 0$ , then the energy coincides with the generalised energy**

This is the generic situation that we have if the transformation between Cartesian and generalised coordinates is time independent and thus  $\frac{\partial \mathbf{r}_i}{\partial t} = 0$ . (Also recall that  $U$  is independent of  $\dot{\mathbf{q}}$ .)

In a practical example we would first check whether  $L$  depends on  $t$  or not. If it is independent,  $h$  is conserved. If  $T$  is quadratic in  $\dot{\mathbf{q}}$  we simply have  $h = E$ , otherwise we need to calculate  $h$  explicitly.

**Example where these conditions are satisfied:**

For the inclined plane we had

$$L = T - U = \left( \frac{1}{2}(\tilde{M} + m)\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s}\cos\alpha \right) - mgs\sin\alpha .$$

(Here I renamed the mass of the plane into  $\tilde{M}$  to avoid confusion with the matrix  $M$ ).  $L$  is independent of  $t$ , hence  $h$  is conserved. All terms in  $T$  involve products of two factors  $\dot{x}$  or  $\dot{s}$ . Therefore the generalised energy coincides with the energy and we have

$$h = E = T + U = \left( \frac{1}{2}(\tilde{M} + m)\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s}\cos\alpha \right) + mgs\sin\alpha .$$

**Examples where these conditions are violated:**

- Consider a **pendulum fixed to a point**  $(0, y(t))$  **that is moving** depending on time with  $y(t) = y_0 \sin\omega t$ . Using  $\theta$  (see Fig. 2.6) as a generalised coordinate, we can write

$$\mathbf{r}(\theta, t) = \begin{pmatrix} 0 \\ y(t) \end{pmatrix} + l \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix},$$

i.e. the potential is

$$U = mg(y(t) - l\cos\theta) .$$

If we take the derivative

$$\dot{\mathbf{r}} = \begin{pmatrix} 0 \\ \dot{y}(t) \end{pmatrix} + l\dot{\theta} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

we get the kinetic energy

$$T = \frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}m(l^2\dot{\theta}^2 + 2l\sin\theta\dot{\theta}\dot{y}(t) + \dot{y}(t)^2)$$

and the Lagrangian

$$L(\theta, \dot{\theta}, t) = T - U = \frac{1}{2}m(l^2\dot{\theta}^2 + 2l\sin\theta\dot{\theta}\dot{y}(t) + \dot{y}(t)^2) - mg(y(t) - l\cos\theta) .$$

$L$  depends explicitly on  $t$ , so  $h$  is not conserved. Moreover the kinetic energy contains terms linear in  $\dot{\theta}$  and independent of  $\dot{\theta}$ . (Note that  $y$  is not a generalised coordinate so  $\dot{\theta}\dot{y}$  and  $\dot{y}^2$  do not count as quadratic terms!) Hence our condition for  $h = E = T + U$  is violated and we have to compute the generalised energy using

$$h = \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L = \frac{1}{2}m(l^2\dot{\theta}^2 - \dot{y}(t)^2) + mg(y(t) - l\cos(\theta)) \neq T + U .$$

- Another example (that will be on problem sheet 4) involves a particle in a **magnetic field**. In this example a term linear in  $\dot{\mathbf{q}}$  arises such that  $h \neq E$ . However,  $L$  is independent of  $t$ , so  $h$  is still conserved.



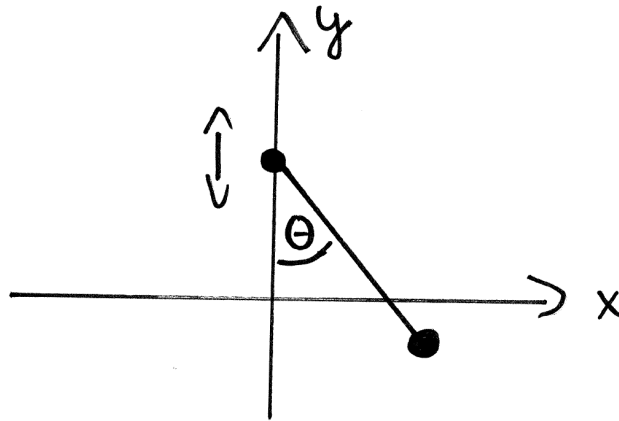


Figure 2.6: A pendulum fixed to a point that is moving.

### 2.4.2 Conservation of generalised momenta

If the Lagrangian is independent of one of the generalised coordinates  $q_\alpha$ , this gives rise to another conservation law. **Lagrange's equations**

$$\frac{\partial L}{\partial q_\alpha} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha}$$

imply:

**Thm:** If  $L$  is independent of one of the generalised coordinates  $q_\alpha$  ( $\frac{\partial L}{\partial q_\alpha} = 0$ ), then

$$p_\alpha \equiv \frac{\partial L}{\partial \dot{q}_\alpha}$$

must be a conserved quantity ( $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} = 0$ ).

Such coordinates not showing up in the Lagrangian are also called **ignorable** or **cyclic** coordinates. The quantity  $p_\alpha$  defined above is called the **generalised momentum** associated to the generalised coordinate  $q_\alpha$ . Hence the statement could also be formulated as follows: **If a coordinate is ignorable, then the corresponding generalised momentum is conserved.**

### Examples

#### a) Conservation of the linear momentum

First of all, let us consider a single particle, described in **Cartesian coordinates**  $x, y, z$ . The Lagrangian of this particle can be written as

$$L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

where  $U(x, y, z)$  is the potential. Now the **generalised momenta**  $p_x$ ,  $p_y$  and  $p_z$  associated to  $x$ ,  $y$  and  $z$  are obtained as

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z}. \end{aligned}$$

We thus see that these generalised momenta just coincide with the **usual (linear) momenta** in  $x$ -,  $y$ - and  $z$ -direction, mass times component of the velocity.

We now obtain the following **conservation law**: If the potential and thus Lagrangian are independent of  $x$ ,

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = 0,$$

then the momentum in  $x$ -direction  $p_x = m\dot{x}$  is conserved. Analogous results hold for  $p_y$  and  $p_z$ .

Within Newtonian mechanics, we would have argued that due to  $\frac{\partial U}{\partial x} = 0$  there is no force in  $x$ -direction and hence the corresponding component of the momentum remains constant.

### b) Angular momentum conservation

Let us now see what happens if we are in the  $x - y$ -plane and instead of Cartesian coordinates pick **polar coordinates**,

$$\mathbf{r} = \rho \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix}.$$

We then have

$$\dot{\mathbf{r}} = \dot{\rho} \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix} + \rho \dot{\phi} \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}$$

and thus

$$\dot{\mathbf{r}}^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2.$$

The Lagrangian reads

$$L = T - U = \frac{1}{2}m(\dot{\rho}^2 + \rho^2 \dot{\phi}^2) - U(\rho, \phi)$$

where the potential was written as a general function of  $\rho$  and  $\phi$ . So what are the **generalised momenta** associated to  $\rho$  and  $\phi$ ? For  $\rho$  we get

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}$$

which is of the same type as the linear momenta, but now with the velocity  $\dot{\rho}$ . Hence  $p_\rho$  can be interpreted as the **radial component of the linear momentum**. The generalised momentum associated to  $\phi$  reads

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2 \dot{\phi}.$$

$p_\phi$  has the interpretation of an **angular momentum**. To check this, we use the definition of the angular momentum  $\mathbf{l} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}}$  from Mechanics 1. Inserting our formula for  $\mathbf{r}$  and  $\dot{\mathbf{r}}$  we obtain

$$\begin{aligned} \mathbf{l} &= \rho \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix} \times \left[ m\dot{\rho} \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix} + m\rho\dot{\phi} \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 \\ 0 \\ m\rho^2\dot{\phi} \end{pmatrix}. \end{aligned}$$

The only interesting component here is the  $z$ -component. (Any cross product of two vectors in the  $x - y$ -plane must point in  $z$ -direction.) As anticipated it coincides with  $p_\phi$ .

Now which conservation law do we get? Since the kinetic energy already depends on  $\rho$ , the radial coordinate cannot be ignorable. But  $\phi$  can be ignorable, if the **potential  $U$  depends only on  $\rho$  and not on  $\phi$** ,

$$\frac{\partial L}{\partial \phi} = -\frac{\partial U}{\partial \phi} = 0.$$

In this case the **angular momentum is conserved**. Such potentials independent of  $\phi$  are also referred to as **central fields**. Central fields are rather common. If we have, e.g., a gravitational field due to a mass at the origin, the potential will only depend on the distance  $\rho$  from the origin.

### c) Inclined plane

For the inclined plane we have

$$L = T - U = \frac{1}{2}(\tilde{M} + m)\dot{x}^2 + \frac{1}{2}m\dot{s}^2 + m\dot{x}\dot{s} \cos \alpha - mgs \sin \alpha.$$

and the generalised momenta are

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = (\tilde{M} + m)\dot{x} + m\dot{s} \cos \alpha \\ p_s &= \frac{\partial L}{\partial \dot{s}} = m\dot{s} + m\dot{x} \cos \alpha. \end{aligned}$$

$L$  is independent of  $x$ , hence  $p_x$  is conserved. But it depends on  $s$ , thus  $p_s$  is not conserved.

### 2.4.3 Spherical pendulum

The spherical pendulum is a good example to illustrate the use of conservation laws in Lagrangian mechanics. It is simply a pendulum that is allowed to move in all three dimensions of space rather than in only two dimensions. To build a spherical pendulum, one simply takes a particle of mass  $m$  and pivots at the origin by a rigid rod of length  $l$ . Since the particle is allowed to move in three-dimensional space but its distance from the origin is fixed to be  $l$ , the possible **particle positions form a sphere of radius  $l$  around the origin**. For simplicity, we will choose units in which  $m$ ,  $g$  and  $l$  all become 1.

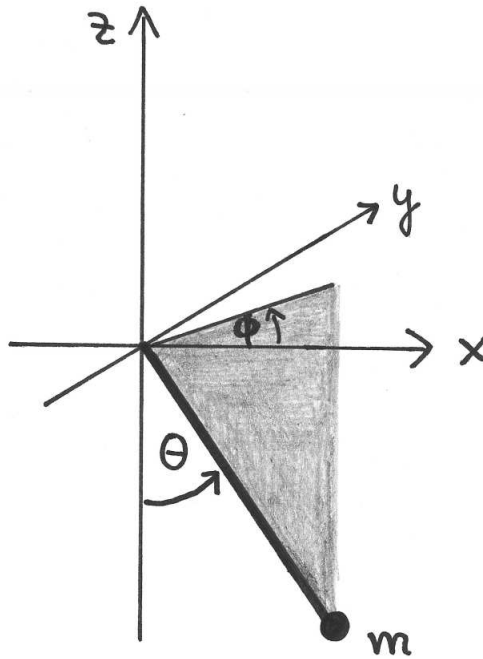


Figure 2.7: Spherical pendulum.

### Find Lagrangian

To look for conservation laws, we first have to find suitable generalised coordinates and write down the Lagrangian. Since the possible particle positions are on a sphere, it is natural to use **spherical coordinates**, but with two twists: First, the radius is fixed to be  $l = 1$ , so there are only two variables  $\theta$  and  $\phi$ . Second, to be consistent with the treatment of the two-dimensional pendulum we define  $\theta$  to be the angle enclosed between the pendulum and the negative  $z$ -axis, not the positive one (see Fig. 2.7) which means that we replace  $\theta$  by  $\pi - \theta$  compared to the usual definition of spherical coordinates. Our modified spherical coordinates thus have the form

$$\mathbf{r} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ -\cos \theta \end{pmatrix}. \quad (2.39)$$

The velocity is now given by

$$\dot{\mathbf{r}} = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix} \dot{\theta} + \begin{pmatrix} -\sin \theta \sin \phi \\ \sin \theta \cos \phi \\ 0 \end{pmatrix} \dot{\phi},$$

and the **kinetic energy** reads

$$T = \frac{1}{2} \dot{\mathbf{r}}^2 = \frac{1}{2} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2).$$

The gravitational **potential** is  $U = mgz$ . Here  $z$  is  $-\cos \theta$  (see(2.39)) and we have set  $m$  and  $g$  to be equal to 1. Thus

$$U = -\cos \theta.$$

The **Lagrangian** therefore has the form

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\sin^2\theta\dot{\phi}^2 + \cos\theta.$$

### Conserved quantities

Now we want to look for conserved quantities, and thus for ignorable coordinates. The Lagrangian  $L$  does not depend on  $\phi$  and  $t$ . This should give rise to two conservation laws:

- Since  $L$  is independent of  $t$ , the generalised energy  $h$  is conserved. Since the kinetic energy is quadratic in the derivatives  $\dot{\theta}$  and  $\dot{\phi}$  the generalised energy also coincides with the energy  $E = T + U$ . Thus we also have **energy conservation**:

$$E = T + U = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\sin^2\theta\dot{\phi}^2 - \cos\theta = \text{const.} \quad (2.40)$$

- Since  $\phi$  is ignorable, the corresponding generalised momentum  $p_\phi$  must be a conserved quantity. We thus have

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \sin^2\theta\dot{\phi} = \text{const.} \quad (2.41)$$

$p_\phi$  can be identified with the  **$z$ -component of the angular momentum** (corresponding to rotations about the  $z$ -axis, which is the symmetry axis of the spherical pendulum). We had already seen that this is the right interpretation of the generalised momentum associated to  $\phi$  in polar coordinates. Similarly, if we use spherical coordinates the  $z$ -component of the angular momentum  $\mathbf{r} \times m\dot{\mathbf{r}}$  (where  $m = 1$ ) is given by

$$\begin{aligned} & xy - y\dot{x} \\ &= \sin\theta\cos\phi(\cos\theta\sin\phi\dot{\theta} + \sin\theta\cos\phi\dot{\phi}) - \sin\theta\sin\phi(\cos\theta\cos\phi\dot{\theta} - \sin\theta\sin\phi\dot{\phi}) \\ &= \sin^2\theta\dot{\phi} \end{aligned}$$

which indeed coincides with  $p_\phi$ .

Conservation laws are so valuable for the description of mechanical systems because they allow us to **reduce the number of independent variables**. In the present case, we can use (2.41) to get rid of  $\dot{\phi}$  in (2.40). If we solve (2.41) for  $\dot{\phi}$  we get

$$\dot{\phi} = \frac{p_\phi}{\sin^2\theta}. \quad (2.42)$$

Inserting this into (2.40) yields

$$E = \frac{1}{2}\dot{\theta}^2 + \frac{p_\phi^2}{2\sin^2\theta} - \cos\theta = \text{const} \quad (2.43)$$

Now we have a differential equation for  $\theta$  only. There is no  $\phi$  because  $T$  and  $U$  are independent of  $\phi$  (which was the reason for angular momentum conservation in the first place), and then  $\dot{\phi}$  could be eliminated using (2.41). Instead we have two constants,  $E$  and  $p_\phi$ .

### Interpretation

Since  $\phi$  is gone, Eq. (2.43) could be interpreted as describing a fictitious **particle in one dimension**, with the only coordinate  $\theta$ . If we assume that this particle feels an effective potential of the form

$$V_{\text{eff}}(\theta) = \frac{p_\phi^2}{2\sin^2\theta} - \cos\theta \quad (2.44)$$

then Eq. (2.43) could be interpreted as an **energy conservation law** for our particle:

$$E = \frac{1}{2}\dot{\theta}^2 + V_{\text{eff}}(\theta) = \text{const} \quad (2.45)$$

Here the effective potential contains both the original potential  $U = -\cos\theta$  and an extra term  $\frac{p_\phi^2}{2\sin^2\theta}$  that originates from the  $\dot{\phi}^2$ -term in the kinetic energy; this term became part of the effective potential due to the elimination of  $\dot{\phi}$ .

We can also take the total time derivative, leading to

$$0 = \frac{dE}{dt} = \dot{\theta}\ddot{\theta} + V'_{\text{eff}}(\theta)\dot{\theta}.$$

Division by  $\dot{\theta}$  then gives

$$\ddot{\theta} = -V'_{\text{eff}}(\theta). \quad (2.46)$$

This means that the acceleration of our fictitious particle is given by the negative derivative of the effective potential, as one would expect. (Recall that we have set the mass equal to one.)

### Method I: Integration

There are two ways to proceed further. The first one (which we won't carry to the end here) is to view (2.45) as a differential equation for  $\theta$ , and try to integrate it. We thus take (2.45) and solve for  $\dot{\theta}$ ,

$$\begin{aligned} \dot{\theta}^2 &= 2(E - V_{\text{eff}}(\theta)) \\ \Rightarrow \dot{\theta} &= \sqrt{2(E - V_{\text{eff}}(\theta))} \end{aligned}$$

We then use separation of variables. We write  $\dot{\theta} = \frac{d\theta}{dt}$  as

$$dt = \frac{d\theta}{\dot{\theta}}$$

and afterwards integrate on both sides. The integral over  $t$  goes from 0 to  $t$ , whereas the integral over  $\theta$  goes from  $\theta(0)$  to  $\theta(t)$ . If we insert the explicit formula for  $\dot{\theta}$  this yields

$$t = \int_0^t dt = \int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\dot{\theta}} = \int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\sqrt{2(E - V_{\text{eff}}(\theta))}}$$

The solution of our problem is thus written as an integral. The evaluation of this integral is rather tricky, and outside the scope of this course. One would finally be lead to so-called elliptic functions, a kind of special functions. One would then have to solve for  $\theta(t)$ , and afterwards determine  $\phi(t)$ .

**Method II: Understand the behaviour of the solution**

An alternative approach is to first understand the **qualitative behaviour** of the motion in the effective potential  $V_{\text{eff}}$ . Then one can use the insight gained to say as much as possible about the solution without having to compute the integral. This is the approach we are going to apply in the following.

**Case  $p_\phi = 0$** 

First let us deal with the simple case that  $p_\phi$  vanishes. According to (2.42) this implies  $\dot{\phi} = 0$ , i.e., the angle  $\phi$  remains constant. This means that the pendulum is only moving in a plane (enclosing an angle  $\phi$  with the  $x$ -axis). The problem is thus reduced to the two-dimensional pendulum of Subsection 2.3.1. Also the equation for  $\theta$  is the same. To see this explicitly, note that for  $p_\phi = 0$  the effective potential (2.44) simply reduces to

$$V_{\text{eff}}(\theta) = -\cos \theta ,$$

which was the gravitational potential. The equation  $\ddot{\theta} = -V'_{\text{eff}}(\theta)$  then boils down to

$$\ddot{\theta} = -\sin \theta$$

coinciding with Eq. (2.6). We thus get the same result as in Subsection 2.3.1, with  $g$ ,  $l$  and  $m$  set equal to unity.

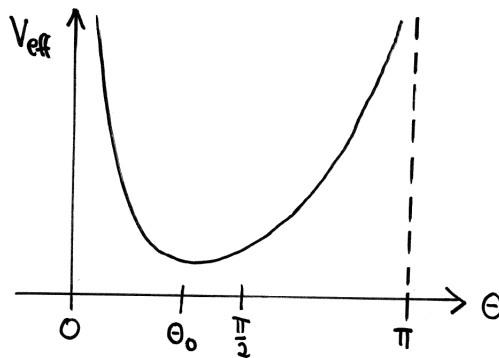
**Case  $p_\phi \neq 0$** 

Figure 2.8: Effective potential of the spherical pendulum.

If  $p_\phi$  is nonzero, the general form of the **effective potential** is given by

$$V_{\text{eff}}(\theta) = \frac{p_\phi^2}{2 \sin^2 \theta} - \cos \theta .$$

It is helpful to visualise the **behaviour of this potential**, see Fig. 2.8. Due to the first summand  $\frac{p_\phi^2}{2 \sin^2 \theta}$  the effective potential **diverges** when  $\sin \theta$  is zero, i.e., for  $\theta \rightarrow 0$  and for  $\theta \rightarrow \pi$ . In both limits,  $V_{\text{eff}}$  tends to  $+\infty$ . Since in spherical coordinates  $\theta$  is restricted to be between 0 and  $\pi$ , it only makes sense to consider  $V_{\text{eff}}(\theta)$  between these two values. So what is the behaviour in between? The simplest possibility

would be that  $V_{\text{eff}}$  just has **one minimum**, and this is what indeed happens. For a proof, let us consider the equation determining the extrema of  $V_{\text{eff}}$ ,

$$0 = V'_{\text{eff}}(\theta_0) = -\frac{p_\phi^2}{\sin^3 \theta_0} \cos \theta_0 + \sin \theta_0 .$$

The effective potential thus becomes extremal for angles  $\theta_0$  with

$$\sin^4 \theta_0 = p_\phi^2 \cos \theta_0 . \quad (2.47)$$

We can show that this equation has only one solution. First let us consider  $0 \leq \theta_0 \leq \pi$  and view the expressions  $\sin^4 \theta_0$  on the l.h.s. and  $p_\phi^2 \cos \theta_0$  on the r.h.s. as functions of  $\theta_0$ . It is easy to see that for  $\theta = 0$  the r.h.s. is larger and for  $\theta = \frac{\pi}{2}$  the l.h.s. is larger. Hence (2.47) must be satisfied somewhere in between. Since the l.h.s.  $\sin^4 \theta_0$  increases monotonically with  $\theta_0$  and the r.h.s.  $p_\phi^2 \cos \theta_0$  decreases monotonically, this can happen only once. So there is only one minimum with  $\theta_0 < \frac{\pi}{2}$ , see Fig. 2.8. For  $\theta_0 > \frac{\pi}{2}$  there can be no solutions since the two sides have opposite signs.

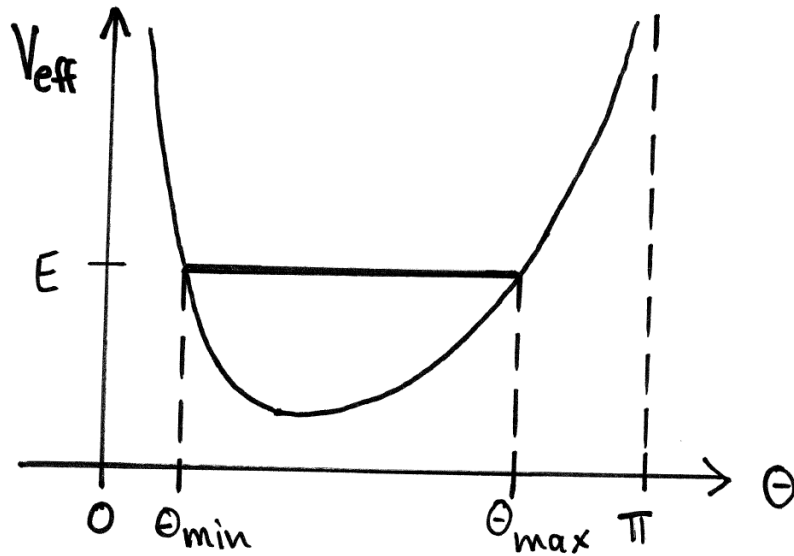


Figure 2.9: Minimal and maximal values of  $\theta$  for the spherical pendulum.

Now what will **solutions** look like? The effective kinetic energy  $\frac{1}{2}\dot{\theta}^2$  must always be positive (or zero).

$$\frac{1}{2}\dot{\theta}^2 = E - V_{\text{eff}}(\theta) \geq 0 .$$

Hence the total energy must always be larger than the effective potential  $V_{\text{eff}}$ . This restricts the possible values of  $\theta$ : For each  $E$  we may only have values of  $\theta$  with  $V_{\text{eff}}(\theta) \leq E$ . Since the effective potential becomes large for  $\theta$  going to 0 or  $\pi$ , this excludes values of  $\theta$  which are too close 0 or  $\pi$ , whereas values further away are permissible. The minimal and maximal values of  $\theta$  for a given value of  $E$  will be denoted by  $\theta_{\text{min}}$  and  $\theta_{\text{max}}$ . They are determined by

$$V_{\text{eff}}(\theta_{\text{min}}) = V_{\text{eff}}(\theta_{\text{max}}) = E .$$



As illustrated in Fig. 2.9,  $\theta_{\min}$  and  $\theta_{\max}$  can be found graphically from the intersections of the graph of  $V_{\text{eff}}(\theta)$  with a straight line corresponding to the energy  $E$ .

Since the angle  $\theta$  is related to the height  $z$  via  $z = -\cos \theta$ , the points  $\theta_{\min}$  and  $\theta_{\max}$  correspond to the lowest and highest points reached for the energy  $E$ . ( $\theta_{\min}$  gives the lowest point and  $\theta_{\max}$  the highest.) For a trajectory with given energy the angle and height will oscillate between these values. This is indicated by the thick line in Fig. 2.8. For  $\theta = \theta_{\min}$  and  $\theta = \theta_{\max}$  the effective kinetic energy  $\frac{1}{2}\dot{\theta}^2$  is zero the whole energy exists as effective potential energy; this includes both the gravitational potential and the kinetic energy due to the change of  $\phi$ . When moving between these values  $\theta$  changes and thus part of the energy is put into  $\frac{1}{2}\dot{\theta}^2$ .

To visualise the trajectories of the pendulum, let us assume that we are looking at the pendulum from above and project the motion into the  $x$ - $y$ -plane. The distance from the centre is then given by

$$\rho = \sqrt{x^2 + y^2} = \sin \theta .$$

Like  $\theta$  and  $z$ , the distance  $\rho$  oscillates between two values,  $\rho_{\min} = \sin \theta_{\min}$  and  $\rho_{\max} = \sin \theta_{\max}$ . For each  $E$  the possible positions are thus enclosed between two circles with these radii. The trajectories must hit the inner circle, then the outer circle, then the inner circle again, etc.

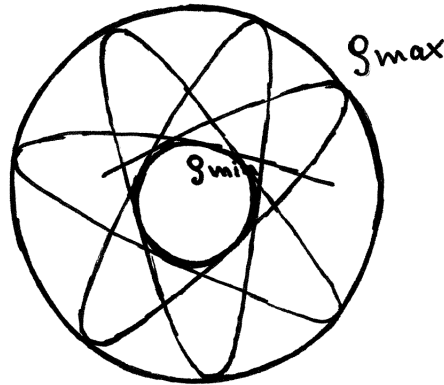


Figure 2.10: Trajectories of the spherical pendulum in the  $x$ - $y$ -plane.

Now what about the angle  $\phi$ ? Due to angular momentum conservation,  $\phi$  changes with a derivative

$$\dot{\phi} = \frac{p_{\phi}}{\sin^2 \theta} .$$

This leads to an effective rotation about the centre, while  $\rho$  oscillates between its minimal and maximal value. The results for  $\rho$  and  $\phi$  together lead to trajectories as sketched in Fig. 2.10.

### Mass moving on a circle

An important special case are orbits where  $\theta$  is constant and the mass just moves on a circle around the  $z$ -axis. In these cases the rod traces out a cone which is why these orbits are sometimes called conical orbits. Such solutions are possible only for  $\theta = \theta_0$ , i.e., if we are at the minimum of the effective potential. This is the

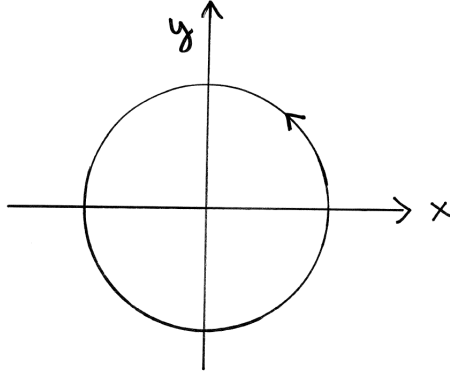


Figure 2.11: Trajectory of the spherical pendulum with  $\theta(t) = \theta_0 = \text{const.}$

only situation in which the acceleration  $\ddot{\theta} = -V'_{\text{eff}}(\theta)$  becomes zero and thus  $\theta$  can remain constant.

Let us now determine how fast the mass must be in this case. For the conical orbits, the angular velocity  $\dot{\phi}$  is given by (see (2.41))

$$\dot{\phi} = \frac{p_{\phi}}{\sin^2 \theta_0} = \text{const.}$$

If we insert Eq. (2.47) and thus  $p_{\phi} = \frac{\sin^2 \theta_0}{\sqrt{\cos \theta_0}}$  we obtain

$$\dot{\phi} = \frac{1}{\sqrt{\cos \theta_0}}.$$

### Mass moving close to a circle

Now what happens if the energy is just slightly above the minimum of the effective energy? In this case the orbits should be close to the conical orbits derived above. But there will be a little bit of energy left that can be invested in going away from the minimum and thus increasing  $V_{\text{eff}}$ , and in changing  $\theta$  and thus having a nonvanishing  $\dot{\theta}$ . We might thus expect that the solutions oscillate about the conical orbits considered above.

To show this we use

$$\ddot{\theta} = -V'_{\text{eff}}(\theta).$$

If we now write  $\theta(t)$  as a sum of the equilibrium value  $\theta_0$  and the deviation from the equilibrium  $\delta\theta$ ,

$$\theta(t) = \theta_0 + \delta\theta.$$

the l.h.s. turns into  $\delta\ddot{\theta}$ . The r.h.s. can be approximated for small  $\delta\theta$  if we make a Taylor expansion

$$V'_{\text{eff}}(\theta) = V'_{\text{eff}}(\theta_0 + \delta\theta) \approx \underbrace{V'_{\text{eff}}(\theta_0)}_{=0} + V''_{\text{eff}}(\theta_0)\delta\theta.$$

If we define  $\Omega_0 = \sqrt{V''_{\text{eff}}(\theta_0)}$  we thus have

$$\delta\ddot{\theta} = -\Omega_0^2\delta\theta.$$

This is the same differential equation as for a harmonic oscillator. The solution involves sines or cosines with a frequency  $\Omega_0$ . In other words:

For orbits close to the conical orbits the angle  $\theta$  of the spherical pendulum oscillates with a frequency  $\Omega_0 = \sqrt{V''_{\text{eff}}(\theta_0)}$ .

(Recall that we have set  $m = 1$ , otherwise there would be a mass here as well.)  
A simple calculation of the second derivative<sup>3</sup> yields

$$\Omega_0 = \sqrt{3 \cos \theta_0 + \frac{1}{\cos \theta_0}}.$$

Due to  $\cos \theta_0 > 0$  this is larger than  $\dot{\phi} = \frac{1}{\sqrt{\cos \theta_0}}$ . Hence the radial oscillations compared to the conical orbits are faster than the oscillation of the angle  $\phi$  due to rotation.

In a more general context the motion close to extrema of a potential (**equilibria**) will be discussed in Chapter 3.

#### 2.4.4 Noether's theorem

So far we have seen that conserved quantities arise when one of the the generalised coordinates  $q_\alpha$  or the time  $t$  don't show up in the Lagrangian. This is an example for a more general principle: In fact all **symmetries** of a system and its Lagrangian give rise to conservation laws. This principle will be very helpful to get counterparts for the conservation of linear and angular momenta (see Subsection 2.4.2) in a system with many particles. We will first study a further example and then generalise.

##### Translation symmetry

Consider a system with  $N$  particles at  $\mathbf{r}_1, \dots, \mathbf{r}_N$ . This system is called **symmetric (or invariant) w.r.t. translations in direction  $\mathbf{d}$**  if the the Lagrangian remains the same when all particles positions are shifted by an arbitrary amount in direction  $\mathbf{d}$ . This means that we have

$$L(\mathbf{r}_1, \dots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N, t) = L(\mathbf{r}_1 + s\mathbf{d}, \dots, \mathbf{r}_N + s\mathbf{d}, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N, t)$$

for all real numbers  $s$ .

**Example:** Consider the two-particle system with the Lagrangian

$$L(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2) = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 + \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \quad (2.48)$$

accounting for the kinetic energy of these particles and the potential due to their gravitational attraction. Both the derivatives  $\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2$  and the difference  $|\mathbf{r}_1 - \mathbf{r}_2|$  remain the same if all positions are increased by  $s\mathbf{d}$ . Hence the Lagrangian is not changed if both particles are moved in direction  $\mathbf{d}$ .

Now our statement is:

---

<sup>3</sup>One has to show that  $V''_{\text{eff}}(\theta) = \frac{3p_\phi^2}{\sin^4 \theta} \cos^2 \theta + \frac{p_\phi^2}{\sin^2 \theta} + \cos \theta$ . The result then follows if we insert  $\theta = \theta_0$  and  $p_\phi^2 = \frac{\sin^4 \theta_0}{\cos \theta_0}$  from Eq. (2.47).

**Thm.:** For a system that is invariant w.r.t. translations in direction  $\mathbf{d}$  the component of the **total linear momentum** of all particles  $\sum_{i=1}^N \mathbf{p}_i$  in direction  $\mathbf{d}$  is a conserved quantity, i.e.,

$$\sum_{i=1}^N \mathbf{p}_i \cdot \mathbf{d} = \text{const.}$$

**Proof:** For such a system  $L(\mathbf{r}_1 + s\mathbf{d}, \dots, \mathbf{r}_N + s\mathbf{d}, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N, t)$  does not depend on  $s$ , which means that the derivative of  $L$  w.r.t.  $s$  vanishes. This is true for all  $s$ , but it will be sufficient to consider the derivative at  $s = 0$ . We then get

$$\begin{aligned} 0 &= \left. \frac{\partial}{\partial s} L(\mathbf{r}_1 + s\mathbf{d}, \dots, \mathbf{r}_N + s\mathbf{d}, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N, t) \right|_{s=0} \\ &= \sum_{i=1}^N \frac{\partial L}{\partial \mathbf{r}_i} \cdot \left. \frac{\partial(\mathbf{r}_i + s\mathbf{d})}{\partial s} \right|_{s=0} \\ &= \sum_{i=1}^N \left( \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{\mathbf{r}}_i}}_{=\mathbf{p}_i} \right) \cdot \mathbf{d} \\ &= \frac{d}{dt} \left( \sum_{i=1}^N \mathbf{p}_i \cdot \mathbf{d} \right). \end{aligned}$$

Here in the second line we used the chain rule. In the third line, we used Lagrange's equations and that  $\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \mathbf{p}_i$ ; moreover the derivative of  $\mathbf{r}_i + s\mathbf{d}$  was evaluated. Finally the total time derivative was written in front. Since this derivative vanishes we obtain the desired result

$$\sum_{i=1}^N \mathbf{p}_i \cdot \mathbf{d} = \text{const.}$$

**Note:** Many systems (e.g. (2.48)) are invariant w.r.t. translations in **all directions**  $\mathbf{d}$ . Then all scalar products involving  $\sum_{i=1}^N \mathbf{p}_i$  vanish, which is only possible if **the total linear momentum**  $\sum_{i=1}^N \mathbf{p}_i$  is conserved.

### Noether's theorem in its general form

We now extend our results to systems with a rather general kind of symmetry: We assume that the Lagrangian remains the same if the generalised coordinates (collected into the vector  $\mathbf{q} = (q_1, \dots, q_d)$ ) are replaced by new values  $\mathbf{Q}(\mathbf{q}, s) = (Q_1(\mathbf{q}, s), \dots, Q_d(\mathbf{q}, s))$ ; these new values depend both on the old ones and on a parameter  $s$  that tells us how much the coordinates have been changed. (In the above example  $s$  was the distance of a shift.) For  $s = 0$  we assume that no change occurred, i.e., the coordinates are still the old ones. For such symmetries Emmy Noether (1882-1935) derived the following conservation law:

### Noether's theorem

Consider a system whose Lagrangian  $L(\mathbf{q}, \dot{\mathbf{q}}, t)$  remains the same after replacing  $\mathbf{q}$  by  $\mathbf{Q}(\mathbf{q}, s)$  (where  $s \in \mathbb{R}$  and  $\mathbf{Q}(\mathbf{q}, s)$  is a mapping with  $\mathbf{Q}(\mathbf{q}, 0) = \mathbf{q}$ ), i.e.,

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = L(\mathbf{Q}, \dot{\mathbf{Q}}, t) . \quad (2.49)$$

For such a system the quantity

$$\sum_{\alpha=1}^d \underbrace{\frac{\partial L}{\partial \dot{q}_\alpha}}_{p_\alpha} \frac{\partial Q_\alpha}{\partial s} \Big|_{s=0} \quad (2.50)$$

is conserved.

**Proof:** According to (2.49)  $L(\mathbf{Q}, \dot{\mathbf{Q}}, t)$  is independent of  $s$ , i.e., its derivative w.r.t.  $s$  vanishes. Again we need this fact only for  $s = 0$  (even though it holds true for all  $s$ ). We get

$$\begin{aligned} 0 &= \frac{\partial}{\partial s} L(\mathbf{Q}, \dot{\mathbf{Q}}, t) \Big|_{s=0} \\ &= \sum_{\alpha=1}^d \left[ \frac{\partial L}{\partial Q_\alpha} \frac{\partial Q_\alpha}{\partial s} + \frac{\partial L}{\partial \dot{Q}_\alpha} \frac{\partial \dot{Q}_\alpha}{\partial s} \right] \Big|_{s=0} \\ &= \sum_{\alpha=1}^d \left[ \frac{\partial L}{\partial q_\alpha} \frac{\partial Q_\alpha}{\partial s} \Big|_{s=0} + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{\partial \dot{Q}_\alpha}{\partial s} \Big|_{s=0} \right] \\ &= \sum_{\alpha=1}^d \left[ \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} \right) \frac{\partial Q_\alpha}{\partial s} \Big|_{s=0} + \frac{\partial L}{\partial \dot{q}_\alpha} \left( \frac{d}{dt} \frac{\partial Q_\alpha}{\partial s} \Big|_{s=0} \right) \right] \\ &= \frac{d}{dt} \sum_{\alpha=1}^d \frac{\partial L}{\partial \dot{q}_\alpha} \frac{\partial Q_\alpha}{\partial s} \Big|_{s=0} \end{aligned}$$

Here we first evaluated the derivative w.r.t.  $s$  using the chain rule. Then  $s = 0$  was invoked to replace  $Q_\alpha$ 's by  $q_\alpha$ 's. In the fourth line Lagrange's equations were used and the derivatives w.r.t.  $t$  and  $s$  acting on  $Q_\alpha$  were interchanged. Finally, the total time derivative was written in the beginning. Since this derivative is zero we obtain the desired result

$$\sum_{\alpha=1}^d \frac{\partial L}{\partial \dot{q}_\alpha} \frac{\partial Q_\alpha}{\partial s} \Big|_{s=0} = \text{const} .$$

### Examples:

#### Ignorable coordinates

Let us first check that Noether's theorem contains the one we obtained in case of ignorable coordinates. If  $L$  is independent of one variable  $q_\beta$ , Eq. (2.49) holds with  $Q_\beta = q_\beta + s$ ,  $Q_\alpha = q_\alpha$  for all  $\alpha \neq \beta$ . Then the constant above is

$$\sum_{\alpha=1}^d \frac{\partial L}{\partial \dot{q}_\alpha} \frac{\partial Q_\alpha}{\partial s} \Big|_{s=0} = \sum_{\alpha=1}^d \frac{\partial L}{\partial \dot{q}_\alpha} \delta_{\alpha\beta} = \frac{\partial L}{\partial \dot{q}_\beta} = p_\beta ,$$

i.e. the generalised momentum associated to  $q_\beta$ .

### Translation symmetry

Next we check that for the case of a system that is invariant w.r.t. translations in direction  $\mathbf{d}$  the conserved quantity (2.50) boils down to the corresponding component of the total linear momentum. Our coordinates  $\mathbf{q}$  are now the positions of the  $N$  particles

$$\mathbf{q} = (\mathbf{r}_1, \dots, \mathbf{r}_N)$$

and the Lagrangian remains the same if these coordinates are replaced by

$$\mathbf{Q}(\mathbf{q}, s) = (\mathbf{r}_1 + s\mathbf{d}, \dots, \mathbf{r}_N + s\mathbf{d}) .$$

The conserved quantity (2.50) thus takes the form

$$\sum_{i=1}^N \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \frac{\partial(\mathbf{r}_i + s\mathbf{d})}{\partial s} \Big|_{s=0} = \sum_{i=1}^N \mathbf{p}_i \cdot \mathbf{d}$$

which is indeed the  $\mathbf{d}$ -component of the total linear momentum.

### Rotation symmetry

Let us now apply Noether's law to systems that can be rotated about an axis (for example the  $z$ -axis) without changing anything.

**Reminder:** If we rotate a vector  $\mathbf{r} = (x, y, z)$  about the  $z$ -axis by an angle  $s$  we obtain a new vector  $\mathbf{r}' = (x', y', z')$  with

$$\begin{aligned} x' &= x \cos s - y \sin s \\ y' &= x \sin s + y \cos s \\ z' &= z . \end{aligned}$$

The first two lines are identical to what one gets when rotating about the origin in two dimensions. They were derived in Linear Algebra & Geometry. The  $z$ -component just has to stay the same when rotating about the  $z$ -axis. In a matrix notation this result can be written as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathcal{R}(s)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

The original and the rotated vectors have the same square

$$\mathbf{r}'^2 = x'^2 + y'^2 + z'^2 = (x \cos s - y \sin s)^2 + (x \sin s + y \cos s)^2 + z^2 = x^2 + y^2 + z^2 = \mathbf{r}^2$$

and thus the same norm

$$|\mathbf{r}'| = |\mathcal{R}(s)\mathbf{r}| = |\mathbf{r}| .$$

Now assume that the Lagrangian of a many-particle system is **invariant w.r.t. rotation of all particles about the  $z$ -axis**, i.e.,

$$L(\underbrace{\mathbf{r}_1, \dots, \mathbf{r}_N}_{=\mathbf{q}}, \underbrace{\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N}_{=\dot{\mathbf{q}}}, t) = L(\underbrace{\mathcal{R}(s)\mathbf{r}_1, \dots, \mathcal{R}(s)\mathbf{r}_N}_{=\mathbf{Q}}, \underbrace{\mathcal{R}(s)\dot{\mathbf{r}}_1, \dots, \mathcal{R}(s)\dot{\mathbf{r}}_N}_{=\dot{\mathbf{Q}}}, t).$$

**Example:** Just consider the two-particle system with gravity (2.48) again. Rotation by an angle  $s$  replaces  $\mathbf{r}_1$  by  $\mathcal{R}\mathbf{r}_1$  and  $\mathbf{r}_2$  by  $\mathcal{R}(s)\mathbf{r}_2$ . Due to

$$\begin{aligned} \left(\frac{d}{dt}\mathcal{R}(s)\mathbf{r}_1\right)^2 &= (\mathcal{R}(s)\dot{\mathbf{r}}_1)^2 = \dot{\mathbf{r}}_1^2 \\ \left(\frac{d}{dt}\mathcal{R}(s)\mathbf{r}_2\right)^2 &= (\mathcal{R}(s)\dot{\mathbf{r}}_2)^2 = \dot{\mathbf{r}}_2^2 \\ |\mathcal{R}(s)\mathbf{r}_1 - \mathcal{R}(s)\mathbf{r}_2| &= |\mathcal{R}(s)(\mathbf{r}_1 - \mathbf{r}_2)| = |\mathbf{r}_1 - \mathbf{r}_2| \end{aligned}$$

the Lagrangian remains the same.

**Application of Noether's theorem:** For such systems Noether's theorem gives the conserved quantity

$$C = \sum_{i=1}^N \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \frac{\partial(\mathcal{R}(s)\mathbf{r}_i)}{\partial s} \Big|_{s=0} = \sum_{i=1}^N \mathbf{p}_i \cdot \frac{\partial \mathcal{R}(s)}{\partial s} \Big|_{s=0} \mathbf{r}_i$$

To evaluate  $C$  we simply have to differentiate the rotation matrix w.r.t.  $s$ , to get

$$\frac{\partial \mathcal{R}(s)}{\partial s} \Big|_{s=0} = \begin{pmatrix} -\sin s & -\cos s & 0 \\ \cos s & -\sin s & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big|_{s=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we write the vectors in components  $\mathbf{p}_i = (p_{i1}, p_{i2}, p_{i3})$ ,  $\mathbf{r}_i = (r_{i1}, r_{i2}, r_{i3})$  we obtain

$$\begin{aligned} C &= \sum_{i=1}^N \begin{pmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_{i1} \\ r_{i2} \\ r_{i3} \end{pmatrix} \\ &= \sum_{i=1}^N \begin{pmatrix} p_{i1} \\ p_{i2} \\ p_{i3} \end{pmatrix} \cdot \begin{pmatrix} -r_{i2} \\ r_{i1} \\ 0 \end{pmatrix} \\ &= \sum_{i=1}^N (r_{i1}p_{i2} - r_{i2}p_{i1}). \end{aligned}$$

This is just the  $z$ -component of the sum of angular momenta  $\sum_i \mathbf{r}_i \times \mathbf{p}_i$ , i.e.

$$C = \left( \sum_{i=1}^N \mathbf{r}_i \times \mathbf{p}_i \right) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus our result is:

For a system whose Lagrangian is invariant w.r.t. rotations about the  $z$ -axis the  **$z$ -component of the total angular momentum  $\sum_{i=1}^N \mathbf{r}_i \times \mathbf{p}_i$  is conserved.**

Analogous results hold for the  $y$ -axis, the  $z$ -axis and all other axes going through the origin. Therefore:

For a system whose Lagrangian is invariant w.r.t. rotations about **all axis** going through the origin we have

$$\sum_{i=1}^N \mathbf{r}_i \times \mathbf{p}_i = \text{const.}$$

### Summary

Symmetries of the Lagrangian lead to conservation laws:

symmetry w.r.t.	conserved quantity
translations	total momentum
rotations	total angular momentum

Even energy conservation can be placed in this context: If the Lagrangian does not depend on time this means that the system looks the same for all times, i.e., it is invariant w.r.t. “translations in time”. The resulting conservation law is energy conservation:

symmetry w.r.t.	conserved quantity
translations in time	(generalised) energy

These symmetries are satisfied by a large class of systems:

**Thm:** If the Lagrangian of a system depends only

- the squared velocities  $\dot{\mathbf{r}}_i^2$  and
- the distances  $|\mathbf{r}_i - \mathbf{r}_j|$

of the particles, then all symmetries and conservation laws above are satisfied, i.e.:

- translation symmetry  $\Rightarrow$  total momentum conservation
- rotation symmetry  $\Rightarrow$  total angular momentum conservation
- independence from  $t \Rightarrow$  (generalised) energy conservation

**Proof:** (Just generalise what we said about example (2.48).)

- *Translation symmetry:* The squared velocities  $\dot{\mathbf{r}}_i$  are invariant under translations as replacing  $\mathbf{r}_i$  by  $\mathbf{r}_i + s\mathbf{d}$  doesn't change the derivative:

$$\left( \frac{d}{dt}(\mathbf{r}_i + s\mathbf{d}) \right)^2 = \dot{\mathbf{r}}_i^2.$$

The distances  $|\mathbf{r}_i - \mathbf{r}_j|$  are translation invariant as

$$|(\mathbf{r}_i + s\mathbf{d}) - (\mathbf{r}_j + s\mathbf{d})| = |\mathbf{r}_i - \mathbf{r}_j|.$$



- *Rotation symmetry:* The squared velocities are invariant under rotation, i.e. replacing  $\mathbf{r}_i$  by  $\mathcal{R}(s)\mathbf{r}_i$  as

$$\left(\frac{d}{dt}\mathcal{R}(s)\mathbf{r}_i\right)^2 = (\mathcal{R}(s)\dot{\mathbf{r}}_i)^2 = \dot{\mathbf{r}}_i^2.$$

Here we have used that rotation doesn't change the norm or the square of a vector. The distances are invariant under rotation as

$$|\mathcal{R}(s)\mathbf{r}_i - \mathcal{R}(s)\mathbf{r}_j| = |\mathcal{R}(s)(\mathbf{r}_i - \mathbf{r}_j)| = |\mathbf{r}_i - \mathbf{r}_j|$$

where we have used the same result about the norm or the square.

- *Independence from  $t$ :* By assumption  $L$  does not depend explicitly on time, only through  $\dot{\mathbf{r}}_i^2$  and  $|\mathbf{r}_i - \mathbf{r}_j|$ .

**Note:** The total momentum and the total angular momentum each have three components. The generalised energy is just a real number. Hence there are altogether **seven** conserved numbers.

#### Examples:

- The kinetic energy  $\sum_i m_i \dot{\mathbf{r}}_i^2$  depends only on squared velocities. The same applies to many common potentials: e.g. the potential energy of a spring depends only on the distance between the endpoints, the potential due to gravitational attraction between two particles depends only the distance between the particles.
- Assume that our system is in the **gravitational field of the Earth**. Then the potential will contain terms

$$-\frac{Gm_i m_{\text{Earth}}}{|\mathbf{r}_i - \mathbf{r}_{\text{Earth}}|}$$

accounting for the gravitational attraction between each particle  $i$  and the Earth. These terms will satisfy the above conditions only if we take Earth as part of our system. Then we have a so-called **isolated system** where none of the particles interacts with anything outside the system, and all our conservation laws hold.

- Now let's see which conservation laws **if we don't include Earth in our system**, and use the approximation  $m_i g z_i$  for the gravitational potential of the  $i$ -th particle (while all other terms in the Lagrangian are innocent).
  - *Translation symmetry:* The potential remains invariant if we change all  $x_i$  and  $y_i$  by the same amount but not if we change the  $z_i$ 's. Hence we only have translation invariance in  $x$ - and  $y$ -direction, and only the  $x$  and  $y$ -components of the total momentum are conserved.
  - *Rotation symmetry:* The gravitational potential remains invariant if we rotate all particles about the  $z$ -axis, as this does not change their  $z$ -coordinates. Hence the  $z$ -component of the total angular momentum is conserved. However rotations about the  $x$ - or the  $y$ -axis change the  $z$ -coordinates of the particles and thus the gravitational potential. Hence the corresponding components of the total angular momentum are not conserved.

- *Independence from  $t$* : The gravitational potential does not depend explicitly on time. Hence the generalised energy is conserved.

We thus have only **four** conserved numbers.

# Chapter 3

## Small oscillations

### 3.1 General theory

We will now study systems performing *oscillations* close to a minimum of a potential. The prototypical system displaying oscillations is the one-dimensional **harmonic oscillator**, with a potential energy that is quadratic in the coordinate, and a kinetic energy that is quadratic in the velocity. To generalise this example we investigate systems with an arbitrary number  $d$  of generalised coordinates (collected into a vector  $\mathbf{q} = (q_1, q_2, \dots, q_d)$ ) and a Lagrangian of the form:

**Generalised harmonic oscillator**

$$L = T - U = \frac{1}{2} \dot{\mathbf{q}} \cdot M \dot{\mathbf{q}} - \frac{1}{2} \mathbf{q} \cdot K \mathbf{q} . \quad (3.1)$$

Here the potential energy is quadratic in  $\mathbf{q}$  and kinetic energy is quadratic in  $\dot{\mathbf{q}}$ .  $M$  and  $K$  are assumed to be constant, real, symmetric matrices of size  $d \times d$ . Moreover, the matrix  $M$  must be *positive definite*, i.e., for all  $\dot{\mathbf{q}} \neq 0$  we must have  $\dot{\mathbf{q}} \cdot M \dot{\mathbf{q}} > 0$ . This condition is needed because the kinetic energy  $T = \frac{1}{2} \dot{\mathbf{q}} \cdot M \dot{\mathbf{q}}$ , being the sum of the non-negative kinetic energies of the particles making up our system, must be non-negative as well (usually positive, unless all particles are at rest and we thus have  $\dot{\mathbf{q}} = 0$ ).

Applications for Lagrangians of the above type are

- systems composed of springs since the potential of a spring is quadratic.
- Much more generally, we will see that **close to extrema of the potential** (almost) any Lagrangian can be approximated by one of the form (3.1).

#### Lagrange equation

The Lagrangian (3.1) has the derivatives

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{q}} &= -K \mathbf{q} \\ \frac{\partial L}{\partial \dot{\mathbf{q}}} &= M \dot{\mathbf{q}} . \end{aligned}$$

Thus, Lagrange's equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}$$

boils down to

$$M\ddot{\mathbf{q}} = -K\mathbf{q} \quad (3.2)$$

This is a linear differential equation, i.e., linear combinations of solutions are also solutions.

### Normal modes

As for any linear differential equation, it is a good idea to look for particularly simple solutions, and then write general solutions as a linear combination of these simple solutions. Motivated by the example of the harmonic oscillator we thus look for solutions of the form

$$\mathbf{q}(t) = \mathbf{u} \cos(\omega t) \quad (3.3)$$

or

$$\mathbf{q}(t) = \mathbf{u} \sin(\omega t). \quad (3.4)$$

These solutions are called **normal modes**, and  $\omega \in \mathbb{C}$  is called the **normal frequency**. If we insert (3.3) and its second derivative

$$\ddot{\mathbf{q}}(t) = -\mathbf{u} \omega^2 \cos(\omega t)$$

into Lagrange's equation (3.2) we obtain

$$-M\mathbf{u} \omega^2 \cos(\omega t) = -K\mathbf{u} \cos(\omega t).$$

The same result would be obtained for Eq. (3.3) with the cosine replaced by a sine. To have these equations satisfied for all  $t$  we need

$$(K - \omega^2 M)\mathbf{u} = 0. \quad (3.5)$$

This equation is similar to a familiar one: If  $M$  were replaced by the unit matrix  $\mathbf{1}$ , Eq. (3.5) would turn into the equation for eigenvalues and eigenvectors of a matrix  $K$ ,  $(K - \omega^2 \mathbf{1})\mathbf{u} = 0$ . Eq. (3.5) thus represents a **generalised eigenvalue problem**. Accordingly the solutions for  $\omega^2$  are called the **generalised eigenvalues**, and the solutions for  $\mathbf{u}$  are referred to as the **generalised eigenvectors**.

Now how can we get  $\mathbf{u}$  and  $\omega$ ?

- First, we have to realise that  $(K - \omega^2 M)\mathbf{u} = 0$  has nonzero solutions  $\mathbf{u}$  only if

$$\det(K - \omega^2 M) = 0. \quad (3.6)$$

This is because multiplication of a matrix like  $K - \omega^2 M$  with a nonzero vector can yield a zero result only if the determinant of the matrix is equal to zero. Again (3.6) is analogous to the result known from Linear Algebra for the case  $M = \mathbf{1}$ . Eq. (3.6) is called the **characteristic (or secular) equation**.  $\det(K - \omega^2 M)$  is called the **characteristic (or secular) polynomial**. To

show that it is a polynomial and determine its degree we use that like any determinant of a  $d \times d$ -matrix

$$\det(K - \omega^2 M) = \det \begin{pmatrix} K_{11} - \omega^2 M_{11} & K_{12} - \omega^2 M_{12} & \dots \\ K_{21} - \omega^2 M_{21} & K_{22} - \omega^2 M_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

can be written as a sum over products of  $d$  matrix elements. These matrix elements each involve a term proportional to  $\omega^2$  and a term independent of  $\omega^2$ . The product of  $d$  such matrix elements thus contains terms proportional to  $1, \omega^2, \omega^4, \dots$ , up to  $\omega^{2d}$ . Hence  $\det(K - \omega^2 M)$  is a **polynomial of degree  $d$  in  $\omega^2$** . The solutions for  $\omega^2$  in  $\det(K - \omega^2 M) = 0$ , i.e., the generalised eigenvalues, can now be obtained as roots of the characteristic polynomial. As for any polynomial of degree  $d$  there must be  $d$  complex roots. These roots will be denoted by  $\omega_1^2, \omega_2^2, \dots, \omega_d^2$ .

- The generalised eigenvectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d$  corresponding to  $\omega_1^2, \omega_2^2, \dots, \omega_d^2$  are now obtained as solutions of the equation

$$(K - \omega_j^2 M)\mathbf{u}_j = 0. \quad (3.7)$$

We will focus on the case where all eigenvalues  $\omega_j^2$  are different from each other. Then there is one generalised eigenvector  $\mathbf{u}_j$  for each generalised eigenvalue (up to multiplication with a complex number).<sup>1</sup>

If a generalised eigenvalue  $\omega_j^2$  is zero, the corresponding normal modes  $\mathbf{u}_j \cos(\omega_j t)$  and  $\mathbf{u}_j \sin(\omega_j t)$  have to be replaced by  $\mathbf{u}_j$  and  $\mathbf{u}_j t$ . This can be seen as follows:  $\mathbf{q}(t) = \mathbf{u}_j$  is a solution because insertion into (3.2) leads to

$$M \frac{d^2 \mathbf{u}_j}{dt^2} = -K \mathbf{u}_j \iff 0 = -K \mathbf{u}_j$$

which is satisfied due to  $(K - \omega_1^2 M)\mathbf{u}_j$  with  $\omega_1^2 = 0$ . Similarly for  $\mathbf{q}(t) = \mathbf{u}_j t$  we get

$$M \frac{d^2 \mathbf{u}_j}{dt^2} = -K \mathbf{u}_j t \iff 0 = -K \mathbf{u}_j t.$$

### General solution

The general solution of our equations of motion (3.2) can be written as a linear combination of the normal modes. If all eigenvalues are different from zero we thus obtain

$$\mathbf{q}(t) = \sum_{j=1}^d (a_j \mathbf{u}_j \cos(\omega_j t) + b_j \mathbf{u}_j \sin(\omega_j t)) \quad (3.8)$$

where the constants  $a_j, b_j$  are determined by the initial conditions.

<sup>1</sup>The characteristic equation may also have multiple roots. For example, if  $\det(K - \omega^2 M)$  is proportional to  $(\omega^2 - 1)^3$  then  $\omega^2 = 1$  is a triple root, and three of the eigenvalues, say,  $\omega_1^2, \omega_2^2$  and  $\omega_3^2$ , coincide. One can show that for an  $n$ -fold root, the equation  $(K - \omega_j^2 M)\mathbf{u} = 0$  has  $n$  linearly independent solutions (proof: see *Anton, Elementary Linear Algebra* or *Lang, Linear Algebra*). In the above example, we can thus pick linearly independent vectors for  $\mathbf{u}_1, \mathbf{u}_2$ , and  $\mathbf{u}_3$  and then the remaining treatment is the same as in the case where all eigenvalues are different.

**Remarks:**1. **Which values can  $\omega_j$  take?**

First of all, we show that the **generalised eigenvalues  $\omega_j^2$  are always real.**

**Proof:** Take the generalised eigenvalue equation (3.7) and multiply with the complex conjugate of  $\mathbf{u}_j$ , i.e.,  $\mathbf{u}_j^*$ . This gives

$$\mathbf{u}_j^* \cdot (K - \omega_j^2 M) \mathbf{u}_j = 0$$

and thus

$$\mathbf{u}_j^* \cdot K \mathbf{u}_j = \omega_j^2 \mathbf{u}_j^* \cdot M \mathbf{u}_j \Rightarrow \omega_j^2 = \frac{\mathbf{u}_j^* \cdot K \mathbf{u}_j}{\mathbf{u}_j^* \cdot M \mathbf{u}_j}. \quad (3.9)$$

Here  $\mathbf{u}_j^* \cdot K \mathbf{u}_j$  satisfies

$$(\mathbf{u}_j^* \cdot K \mathbf{u}_j)^* = \mathbf{u}_j \cdot K^* \mathbf{u}_j^* = \mathbf{u}_j \cdot K \mathbf{u}_j^* = \mathbf{u}_j^* \cdot K \mathbf{u}_j;$$

the second equality sign follows because  $K$  is real and the third one follows because  $K$  is symmetric.  $\mathbf{u}_j^* \cdot K \mathbf{u}_j$  thus coincides with its complex conjugate, and is a real number. The same applies to  $\mathbf{u}_j^* \cdot M \mathbf{u}_j$ . Due to Eq. (3.9) this means that  $\omega_j^2$  is real.

This leaves the following **possibilities for  $\omega_j$** :

- $\omega_j$  can be **real** and different from zero. Then the normal modes  $\mathbf{u}_j \cos(\omega_j t)$  and  $\mathbf{u}_j \sin(\omega_j t)$  describe **oscillations**. Here the sign of  $\omega_j$  is not important. Changing the sign only flips the sign of  $\mathbf{u}_j \sin(\omega_j t)$  which can be compensated by also flipping the sign of  $b_j$  in our general solution (3.8). Hence in the present case  $\omega_j$  can always be taken positive.
- $\omega_j$  can be **purely imaginary** and different from zero, i.e.  $\omega_j = i\rho_j$  where  $\rho_j$  is real and different from zero. The normal modes then take the form  $\mathbf{u}_j \cos(\omega_j t) = \mathbf{u}_j \cosh(\rho_j t)$  and  $\mathbf{u}_j \sin(\omega_j t) = \mathbf{u}_j \sinh(\rho_j t)$ . Alternatively we can write the solutions as linear combinations of  $\mathbf{u}_j e^{\rho_j t}$  and  $\mathbf{u}_j e^{-\rho_j t}$ , i.e., our coordinates increase or decrease **exponentially**. Similarly as above  $\rho_j$  can always be taken positive.
- We have already shown that  $\omega_j = 0$  leads to normal modes constant and linear in  $t$ .

Due to  $\omega_j^2$  being real the generalised eigenvalue equation  $(K - \omega_j^2 M) \mathbf{u}_j = 0$  is a real equation, and the **eigenvectors  $\mathbf{u}_j$  can be chosen real** as well.

2. **Are the eigenvectors orthogonal?**

For  $M = 1$ , it was shown in Linear Algebra that the eigenvectors corresponding to different eigenvalues are orthogonal, i.e.,  $\mathbf{u}_j \cdot \mathbf{u}_k = 0$  if  $\omega_j^2 \neq \omega_k^2$ . For general  $M$  we obtain a **generalised orthogonality relation**: For  $\omega_j^2 \neq \omega_k^2$  we have  $\mathbf{u}_j \cdot M \mathbf{u}_k = 0$ .

**Proof:** We write

$$\begin{aligned} (\omega_j^2 - \omega_k^2) \mathbf{u}_j \cdot M \mathbf{u}_k &= \mathbf{u}_k \cdot \omega_j^2 M \mathbf{u}_j - \mathbf{u}_j \cdot \omega_k^2 M \mathbf{u}_k \\ &= \mathbf{u}_k \cdot K \mathbf{u}_j - \mathbf{u}_j \cdot K \mathbf{u}_k \\ &= 0. \end{aligned}$$

In the first line, the symmetry of  $M$  was used to replace one  $\mathbf{u}_j \cdot M\mathbf{u}_k$  by  $\mathbf{u}_k \cdot M\mathbf{u}_j$ . In the second line, the eigenvector equation  $\omega_j^2 M\mathbf{u}_j = K\mathbf{u}_j$  (similarly for  $k$ ) was invoked. The third line follows from the symmetry of  $K$ .

If we normalise the eigenvectors according to  $\mathbf{u}_j \cdot M\mathbf{u}_j = 1$ , and all generalised eigenvalues are different, we even have

$$\mathbf{u}_j \cdot M\mathbf{u}_k = \delta_{jk}.$$

This generalises the relation  $\mathbf{u}_j \cdot \mathbf{u}_k = \delta_{jk}$  for the eigenvector problem studied in Linear Algebra.

### 3. Are the eigenvectors linearly independent?

Suppose that the generalised eigenvalues  $\omega_1^2, \omega_2^2, \dots, \omega_d^2$  are distinct. Then **the generalised eigenvectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d$  are linearly independent.**

**Proof:** We have to show that 0 cannot be written as a nontrivial linear combination of the eigenvectors  $\mathbf{u}_j$ , i.e.,  $\sum_{j=1}^d c_j \mathbf{u}_j = 0$  implies that all coefficients  $c_j$  are 0. Let us thus assume that  $\sum_{j=1}^d c_j \mathbf{u}_j = 0$ . Multiplication with  $\mathbf{u}_k$  and  $M$  leads to

$$\sum_{j=1}^d c_j \mathbf{u}_k \cdot M\mathbf{u}_j = 0. \quad (3.10)$$

According to 3. the product  $\mathbf{u}_k \cdot M\mathbf{u}_j$  is equal to zero for all  $j \neq k$ . Therefore the sum in (3.10) only receives a contribution from  $j = k$ , and we obtain

$$c_k \mathbf{u}_k \cdot M\mathbf{u}_k = 0. \quad (3.11)$$

Since  $M$  is positive definite we have  $\mathbf{u}_k \cdot M\mathbf{u}_k \neq 0$  and thus (3.11) can only be satisfied if

$$c_k = 0$$

for all  $k$ , as claimed.

## 3.2 Two springs

Let us apply the ideas above to a system of two springs, connecting three freely moving blocks of mass  $m_1$ ,  $m_2$  and  $m_1$ . Both springs have the spring constant  $k$  and the natural length  $l$ . If the blocks are located at positions  $-l$ ,  $0$  and  $l$  as below the springs have their natural length and the potential is zero:

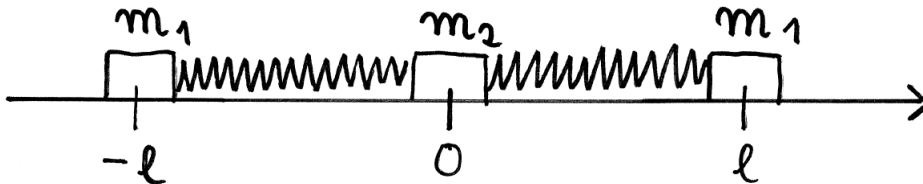


Figure 3.1: A system of two springs connecting masses at  $-l$ ,  $0$  and  $l$ .

We now assume that the blocks are displaced from these positions by  $q_1$ ,  $q_2$  and  $q_3$ , and take  $q_1$ ,  $q_2$  and  $q_3$  as our generalised coordinates: The **kinetic energy** can

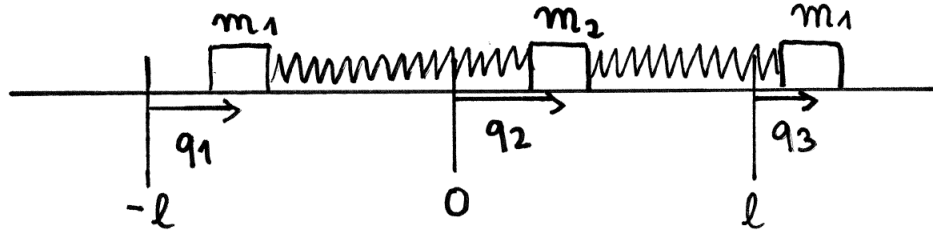


Figure 3.2: A system of two springs connecting masses at  $-l + q_1$ ,  $q_2$  and  $l + q_3$ .

then be written as

$$T = \frac{m_1}{2} \dot{q}_1^2 + \frac{m_2}{2} \dot{q}_2^2 + \frac{m_1}{2} \dot{q}_3^2.$$

In matrix notation  $T$  reads

$$T = \frac{1}{2} \dot{\mathbf{q}} \cdot M \dot{\mathbf{q}}$$

with the mass matrix

$$M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 \end{pmatrix}.$$

The **potential energy** of each spring is  $\frac{k}{2}$  times the square of the displacement from the natural length. Since the length of the first spring differs from the natural length by  $q_2 - q_1$ , and the length of the second spring by  $q_3 - q_2$ , we obtain

$$U = \frac{k}{2} (q_2 - q_1)^2 + \frac{k}{2} (q_3 - q_2)^2 = \frac{k}{2} (q_1^2 + 2q_2^2 + q_3^2 - 2q_1q_2 - 2q_2q_3)$$

In matrix notation we thus have

$$U = \frac{1}{2} \mathbf{q} \cdot K \mathbf{q}$$

where

$$K = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

(Note that when determining the entries of  $K$ , it is important that there is one (diagonal) term in  $K$  corresponding to for each of the squares  $q_1^2$ ,  $q_2^2$ ,  $q_3^2$ , but two entries corresponding to each of the mixed terms  $q_1q_2$ ,  $q_2q_3$ ; hence the coefficients of  $q_1q_2$  and  $q_2q_3$  have to be divided by two.)

With the  $K$  and  $M$  thus derived the **secular equation** reads

$$\begin{aligned} 0 &= \det(K - \omega^2 M) \\ &= \det \begin{pmatrix} k - \omega^2 m_1 & -k & 0 \\ -k & 2k - \omega^2 m_2 & -k \\ 0 & -k & k - \omega^2 m_1 \end{pmatrix} \\ &= (k - \omega^2 m_1)^2 (2k - \omega^2 m_2) - 2k^2 (k - \omega^2 m_1) \\ &= (k - \omega^2 m_1) (2k^2 - k\omega^2 m_2 - 2k\omega^2 m_1 + \omega^4 m_1 m_2 - 2k^2) \\ &= (k - \omega^2 m_1) \omega^2 (m_1 m_2 \omega^2 - km_2 - 2km_1) \end{aligned}$$



and we obtain the following **generalised eigenvalues**

$$\begin{aligned}\omega_1^2 &= 0 \\ \omega_2^2 &= \frac{k}{m_1} \\ \omega_3^2 &= \frac{k}{m_1} + \frac{2k}{m_2}\end{aligned}$$

For each  $\omega_j^2$  the corresponding **generalised eigenvector** is obtained from

$$0 = (K - \omega_j^2 M)\mathbf{u}_j = \begin{pmatrix} k - \omega_j^2 m_1 & -k & 0 \\ -k & 2k - \omega_j^2 m_2 & -k \\ 0 & -k & k - \omega_j^2 m_1 \end{pmatrix} \mathbf{u}_j$$

We thus get:

- for  $\omega_1^2 = 0$

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \mathbf{u}_1 = 0 \implies \mathbf{u}_1 \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(here  $\propto$  means “proportional to”)

- for  $\omega_2^2 = \frac{k}{m_1}$

$$\begin{pmatrix} 0 & -k & 0 \\ -k & (2 - \frac{m_2}{m_1})k & -k \\ 0 & -k & 0 \end{pmatrix} \mathbf{u}_2 = 0 \implies \mathbf{u}_2 \propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

- for  $\omega_3^2 = \frac{k}{m_1} + \frac{2k}{m_2}$

$$\begin{pmatrix} -2\frac{m_1}{m_2} & -k & 0 \\ -k & -\frac{m_1}{m_2}k & -k \\ 0 & -k & -2\frac{m_1}{m_2}k \end{pmatrix} \mathbf{u}_3 = 0 \implies \mathbf{u}_3 \propto \begin{pmatrix} 1 \\ -2\frac{m_1}{m_2} \\ 1 \end{pmatrix}$$

The **general solution** is a linear combination of the normal modes (sketched in Fig. 3.3), i.e.,

$$\mathbf{q}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (a_1 + b_1 t) + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} (a_2 \cos \omega_2 t + b_2 \sin \omega_2 t) + \begin{pmatrix} 1 \\ -2\frac{m_1}{m_2} \\ 1 \end{pmatrix} (a_3 \cos \omega_3 t + b_3 \sin \omega_3 t)$$

The form of the first normal mode arise because no outside forces act on the system (apart from gravity which is compensated by a force of constraint). Hence all particles may be translated by the same amount or move with the same constant velocity without changing anything. In the second mode the inner mass remains fixed while the two outer masses oscillate with opposite phase. In the third mode all masses oscillate, the directions of oscillation of the outer masses coincide and are opposite to the direction of oscillation of the inner mass.

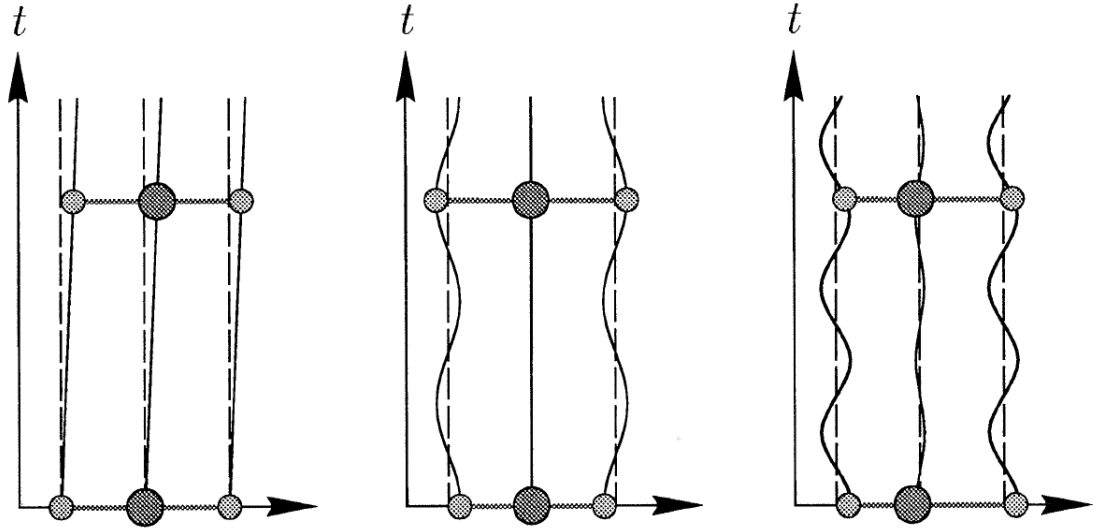


Figure 3.3: Normal modes of a system with two springs, corresponding to the generalised eigenvectors  $\omega_1^2$ ,  $\omega_2^2$  and  $\omega_3^2$ .

### 3.3 Small oscillations about equilibrium

We now come to the main application of the generalised harmonic oscillator (3.1): We will consider a rather large class of systems, namely those with an arbitrary potential and a kinetic energy that is quadratic in  $\dot{\mathbf{q}}$ ,

$$L(\mathbf{q}, \dot{\mathbf{q}}) = T - U = \frac{1}{2} \dot{\mathbf{q}} \cdot M(\mathbf{q}) \dot{\mathbf{q}} - U(\mathbf{q}) = \frac{1}{2} \sum_{\alpha=1}^d \sum_{\beta=1}^d \dot{q}_\alpha M_{\alpha\beta}(\mathbf{q}) \dot{q}_\beta - U(\mathbf{q}). \quad (3.12)$$

(Here  $M(\mathbf{q})$  is again a real symmetric, positive definite matrix.) We will show that close to stationary points of the potential the Lagrangian of these systems can be approximated by a Lagrangian of the form (3.1). We will then apply the theory of normal modes to the motion close to these points.

#### Equilibria

Points  $\mathbf{q}_*$  where the potential is stationary are called **equilibrium points**. At these points the derivatives of  $U$  must vanish,

$$\frac{\partial U}{\partial \mathbf{q}}(\mathbf{q}_*) = 0.$$

It is important that **particles can stay fixed at equilibrium points**. This is immediately clear in Newtonian mechanics: The (conservative) forces are given by derivatives of the potential. At equilibrium points these derivatives vanish and thus there are no forces. Hence particles may rest at these points.

The **proof** within Lagrangian mechanics looks as follows: With the Lagrangian in (3.12) Lagrange's equations  $\frac{\partial L}{\partial q_\gamma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\gamma}$  take the form

$$\frac{1}{2} \sum_{\alpha,\beta=1}^d \dot{q}_\alpha \frac{\partial M_{\alpha\beta}}{\partial q_\gamma} \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma} = \frac{d}{dt} \sum_{\beta=1}^d M_{\gamma\beta} \dot{q}_\beta = \sum_{\beta=1}^d \left( \frac{dM_{\gamma\beta}}{dt} \dot{q}_\beta + M_{\gamma\beta} \ddot{q}_\beta \right). \quad (3.13)$$

This equation is indeed satisfied for  $\mathbf{q}(t) = \mathbf{q}_* = \text{const}$ , since for constant  $\mathbf{q}$  the time derivatives vanish, and the term  $\frac{\partial U}{\partial q_\gamma}$  vanishes at stationary points of  $U$ .

### Approximation close to equilibrium

To give an approximation for the Lagrangian  $L$  that is valid close to the equilibrium, we write

$$\mathbf{q}(t) = \mathbf{q}_* + \delta\mathbf{q}(t)$$

which in particular implies

$$\dot{\mathbf{q}} = \delta\dot{\mathbf{q}}.$$

We then make a Taylor expansion up to second order in  $\delta\mathbf{q}$ ,  $\delta\dot{\mathbf{q}}$ .

For the **potential** this expansion reads

$$U(\mathbf{q}) \approx U(\mathbf{q}_*) + \sum_{\alpha=1}^d \underbrace{\frac{\partial U}{\partial q_\alpha}(\mathbf{q}_*)}_{=0} \delta q_\alpha + \frac{1}{2} \sum_{\alpha=1}^d \sum_{\beta=1}^d \underbrace{\frac{\partial^2 U}{\partial q_\alpha \partial q_\beta}(\mathbf{q}_*)}_{\equiv K_{\alpha\beta}} \delta q_\alpha \delta q_\beta$$

where  $\approx$  indicates dropping all terms of cubic or higher order. The linear term vanishes due to  $\frac{dU}{dq_\alpha}(\mathbf{q}_*) = 0$ . The quadratic terms can be written in a simple form if we collect the second derivatives of the potential at the equilibrium point into a matrix  $K$  with entries

$$K_{\alpha\beta} = \frac{\partial^2 U}{\partial q_\alpha \partial q_\beta}(\mathbf{q}_*).$$

We then obtain

$$U(\mathbf{q}) \approx U(\mathbf{q}_*) + \frac{1}{2} \delta\mathbf{q} \cdot K \delta\mathbf{q} \quad (3.14)$$

For the **kinetic energy** we write

$$\begin{aligned} T &= \frac{1}{2} \mathbf{q} \cdot M(\mathbf{q}) \mathbf{q} = \frac{1}{2} \delta\dot{\mathbf{q}} \cdot M(\mathbf{q}) \delta\dot{\mathbf{q}} \\ &\approx \frac{1}{2} \delta\dot{\mathbf{q}} \cdot (M(\mathbf{q}_*) + \text{terms of order } \delta\mathbf{q} \text{ and higher}) \delta\dot{\mathbf{q}}. \end{aligned}$$

Since we drop all terms with more than two  $\delta\mathbf{q}$ ,  $\delta\dot{\mathbf{q}}$  this is approximated by

$$T \approx \frac{1}{2} \delta\dot{\mathbf{q}} \cdot M(\mathbf{q}_*) \delta\dot{\mathbf{q}}.$$

In the vicinity of the equilibrium point  $\mathbf{q}_*$ , the Lagrangian  $L$  thus becomes

$$L = T - U \approx \frac{1}{2} \delta\dot{\mathbf{q}} \cdot M(\mathbf{q}_*) \delta\dot{\mathbf{q}} - U(\mathbf{q}_*) - \frac{1}{2} \delta\mathbf{q} \cdot K \delta\mathbf{q} \quad (3.15)$$

This is just the form of the Lagrangian considered in Section (3.1), with  $\delta\mathbf{q}$  and  $\delta\dot{\mathbf{q}}$  taking the place of  $\mathbf{q}$  and  $\dot{\mathbf{q}}$ . The only difference is the constant  $-U(\mathbf{q}_*)$  which however does not affect the equations of motion.

**Example:** Imagine a linear molecule with three atoms of mass  $m_1$ ,  $m_2$  and  $m_3$ , and a potential that consists of a complicated term depending on the distance between the first and the second atom, and an analogous term depending on the distance between the second and the third atom. Apart from the form of the potential this situation is analogous to the system with two springs in Section 3.2

If we are close to equilibrium, a quadratic approximation of this potential gives rise to the same Lagrangian as for the two-spring system (with a spring constant that depends on the second derivatives of the complicated potential).

### Types of equilibria

We are now prepared to study the motion close to equilibrium. We will see that there is a fundamental difference between equilibria corresponding to minima and maxima of the potential.

- If the potential becomes **minimal** at  $\mathbf{q}_*$ , then  $U(\mathbf{q}) = U(\mathbf{q}_*) + \delta\mathbf{q} \cdot K\delta\mathbf{q}$  (with  $\delta\mathbf{q} \neq 0$ ) must be larger than  $U(\mathbf{q}_*)$ . Hence  $K$  is **positive definite**,

$$\delta\mathbf{q} \cdot K\delta\mathbf{q} > 0 \text{ for all } \delta\mathbf{q} \neq 0.$$

In this case all **normal frequencies**  $\omega_j$  **are real**, i.e., the particles oscillate around the equilibrium position. Such equilibria are called **stable**.

**Proof:** Recall Eq. (3.9) and thus

$$\omega_j^2 = \frac{\mathbf{u}_j \cdot K\mathbf{u}_j}{\mathbf{u}_j \cdot M\mathbf{u}_j}. \quad (3.16)$$

If both  $M$  and  $K$  are positive definite, this implies  $\omega_j^2 > 0$  and thus  $\omega_j \in \mathbb{R}$ .

**Intuitive argument:** Conservative forces always point in the direction where the potential decreases – e.g. the gravity points downwards, where the gravitational potential decreases. If the equilibrium corresponds to a minimum of the potential, the force pulls particles back equilibrium. This leads to oscillations around the equilibrium configuration.

**Note:** The approximation of  $L$ , Eq. (3.15) is valid only in the vicinity of the equilibrium. Our theory thus describes correctly only the **small oscillations** about the minimum.

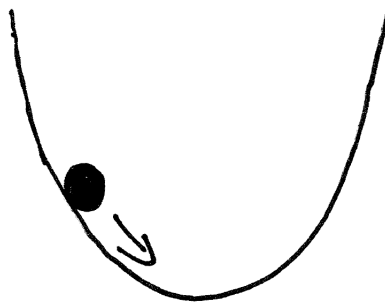


Figure 3.4: Example of a stable equilibrium.

- For a **maximum** of the potential the same argument as above implies that  $K$  is **negative definite**, i.e., we have

$$\delta\mathbf{q} \cdot K\delta\mathbf{q} < 0 \text{ for all } \delta\mathbf{q} \neq 0$$

In this case all **normal frequencies**  $\omega_j$  are **purely imaginary** ( $\omega_j = i\rho_j$ ,  $\rho_j \in \mathbb{R}$ ) and the normal modes are proportional to increasing and decreasing exponentials  $e^{\rho_j t}$ ,  $e^{-\rho_j t}$ . Such equilibria are called **unstable**.

**Proof:** We now have  $\mathbf{u}_j \cdot K\mathbf{u}_j < 0$  and  $\mathbf{u}_j \cdot M\mathbf{u}_j > 0$ . This means that  $\omega_j^2$  must be negative, and  $\omega_j$  must be imaginary.

**Intuitive argument:** The forces again point in the direction where the potential decreases – which this time means away from the equilibrium. Thus particles are usually pushed away from the equilibrium. However if we finely tune the initial velocity a particle approaching the equilibrium may just get slower and slower without ever changing direction or crossing the equilibrium. The latter situation corresponds to the normal modes  $\mathbf{u}_j e^{-\rho_j t}$ .

**Note:** As above our solution is no longer applicable one we are far away from the equilibrium, since then the quadratic approximation of the Lagrangian is no longer valid.

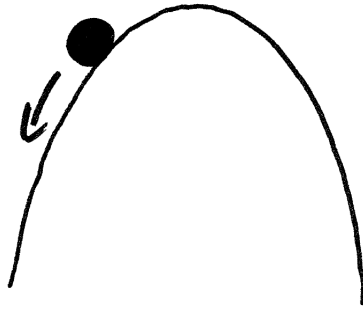


Figure 3.5: Example of an unstable equilibrium.

- If  $K$  is neither positive or negative definite, there can be real, imaginary and zero normal frequencies.

### 3.4 The double pendulum

As an example we consider the double pendulum depicted in Fig. 3.6. We assume that the two masses and the lengths of the two pendulums coincide. The motion of the double pendulum can be very complicated (“chaotic”) in general, but we will see that it becomes simple close to equilibrium. The two masses have positions

$$\mathbf{r}_1 = l \begin{pmatrix} \sin \theta_1 \\ -\cos \theta_1 \end{pmatrix}, \quad \mathbf{r}_2 = l \begin{pmatrix} \sin \theta_1 \\ -\cos \theta_1 \end{pmatrix} + l \begin{pmatrix} \sin \theta_2 \\ -\cos \theta_2 \end{pmatrix}$$

and velocities

$$\dot{\mathbf{r}}_1 = l\dot{\theta}_1 \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix}, \quad \dot{\mathbf{r}}_2 = l\dot{\theta}_1 \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \end{pmatrix} + l\dot{\theta}_2 \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \end{pmatrix}$$

with

$$\dot{\mathbf{r}}_1^2 = l^2 \dot{\theta}_1^2, \quad \dot{\mathbf{r}}_2^2 = l^2 \dot{\theta}_1^2 + 2l^2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + l^2 \dot{\theta}_2^2.$$

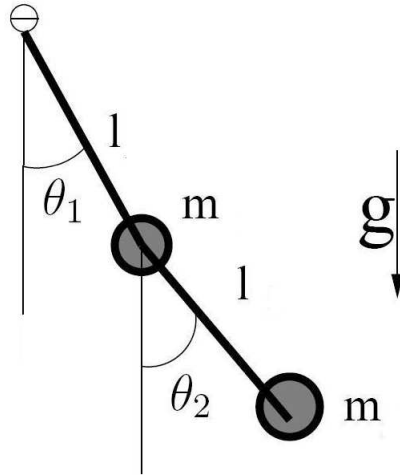


Figure 3.6: The double pendulum.

The kinetic energy of the double pendulum is therefore obtained as

$$T = \frac{1}{2}m(\dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2) = \frac{1}{2}ml^2 \left( 2\dot{\theta}^2 + 2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2 \right).$$

It is helpful to rewrite  $T$  in matrix notation. We then get

$$T = \frac{1}{2}\dot{\mathbf{q}} \cdot M(\mathbf{q})\dot{\mathbf{q}}$$

with  $\mathbf{q} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$  and the mass matrix

$$M(\mathbf{q}) = ml^2 \begin{pmatrix} 2 & \cos(\theta_1 - \theta_2) \\ \cos(\theta_1 - \theta_2) & 1 \end{pmatrix}. \quad (3.17)$$

The **potential energy** is given by

$$U = -2mgl \cos \theta_1 - mgl \cos \theta_2.$$

### Equilibria

To find **equilibria**, we have to set the derivatives of  $U$  w.r.t. the generalised coordinates equal to zero. This leads to

$$\begin{aligned} \frac{\partial U}{\partial \theta_1} = 0 &\Rightarrow \sin \theta_1 = 0 \Rightarrow \theta_1 = 0 \text{ or } \theta_1 = \pi \\ \frac{\partial U}{\partial \theta_2} = 0 &\Rightarrow \sin \theta_2 = 0 \Rightarrow \theta_2 = 0 \text{ or } \theta_2 = \pi. \end{aligned}$$

The four equilibria found in this way are sketched in Fig. 3.7. We would expect the equilibrium at  $\theta_1 = \theta_2 = 0$  to be stable and the one at  $\theta_1 = \theta_2 = \pi$  to be unstable while the two others should be mixed (with one real and one purely imaginary normal frequency). We shall investigate in detail the motion close to the stable and the unstable equilibrium.

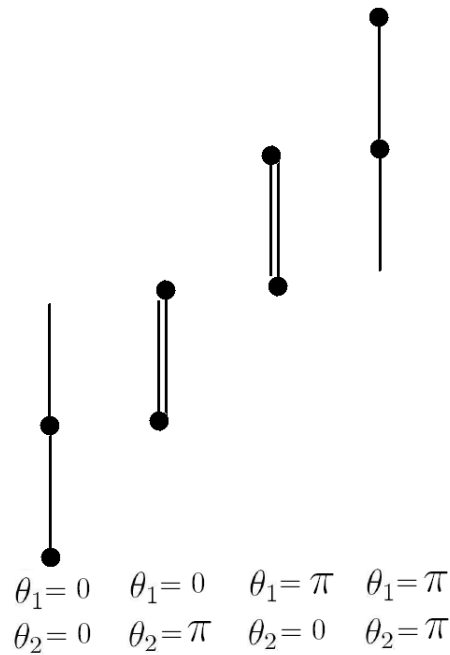


Figure 3.7: Equilibrium positions of the double pendulum.

### Stable equilibrium

Since the first equilibrium is at

$$\mathbf{q}_* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.18)$$

the deviations from equilibrium  $\delta\mathbf{q}$  coincide with the generalised coordinates  $\mathbf{q}$ . The quadratic approximation of the potential is obtained easily if we use the Taylor expansion of the cosine  $\cos\theta = 1 - \frac{1}{2}\theta^2 + \dots$ . We then get

$$\begin{aligned} U &= -2mgl \cos\theta_1 - mgl \cos\theta_2 \\ &\approx -2mgl \left(1 - \frac{\theta_1^2}{2}\right) - mgl \left(1 - \frac{\theta_2^2}{2}\right) \\ &= \frac{1}{2}\delta\mathbf{q} \cdot K\delta\mathbf{q} + \text{const} \end{aligned}$$

with

$$K = mgl \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the quadratic approximation of the **kinetic energy**, we just have to evaluate the mass matrix at the equilibrium position. We then get

$$\begin{aligned} T &\approx \frac{1}{2}\delta\dot{\mathbf{q}} \cdot M(\mathbf{q}_*)\delta\dot{\mathbf{q}} \\ M(\mathbf{q}_*) &= ml^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

The generalised eigenvectors  $\omega^2$  and the normal frequencies  $\omega$  can now be obtained from the **secular equation**

$$\begin{aligned} 0 &= \det(K - \omega^2 M) \\ &= \det ml^2 \begin{pmatrix} 2\frac{g}{l} - 2\omega^2 & -\omega^2 \\ -\omega^2 & \frac{g}{l} - \omega^2 \end{pmatrix} \\ \Rightarrow 0 &= 2\left(\frac{g}{l} - \omega^2\right)^2 - \omega^4 \\ \Rightarrow \omega^2 &= (2 \pm \sqrt{2})\frac{g}{l} > 0 \end{aligned}$$

As expected both normal frequencies are real. The corresponding generalised eigenvectors are easily obtained as  $\mathbf{u} \propto \begin{pmatrix} 1 \\ \mp\sqrt{2} \end{pmatrix}$ . This leads to the two types of motion sketched in Fig. 3.8.

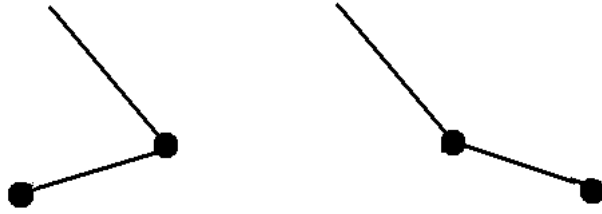


Figure 3.8: Normal modes related to the stable equilibrium of the double pendulum.

### Unstable equilibrium

We now consider the equilibrium at

$$\mathbf{q}_* = \begin{pmatrix} \pi \\ \pi \end{pmatrix}.$$

We thus let

$$\mathbf{q} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \mathbf{q}_* + \delta\mathbf{q} = \begin{pmatrix} \pi + \delta\theta_1 \\ \pi + \delta\theta_2 \end{pmatrix}$$

and approximate up to quadratic order in  $\delta\mathbf{q}$  and its derivative. For the potential energy

$$U = -2mgl \cos \theta_1 - mgl \cos \theta_2$$

Taylor expansion of the cosine gives

$$\cos(\theta_1) = \cos(\pi + \delta\theta_1) = -\cos(\delta\theta_1) \approx -1 + \frac{(\delta\theta_1)^2}{2}$$

(similarly for  $\theta_2$ ) and thus

$$U = \frac{1}{2}\delta\mathbf{q} \cdot K \delta\mathbf{q} + \text{const}$$



with

$$K = -mgl \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The **kinetic energy** is approximated by

$$T \approx \frac{1}{2} \delta \dot{\mathbf{q}} \cdot M(\mathbf{q}_*) \delta \dot{\mathbf{q}}$$

where the mass matrix at equilibrium again reads

$$M(\mathbf{q}_*) = ml^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Compared to the stable equilibrium, only  $K$  has changed sign. Since  $K$  is proportional to  $g$  we can thus copy the results from the stable equilibrium with the replacement  $g \rightarrow -g$ . This yields

$$\omega^2 = -(2 \pm \sqrt{2}) \frac{g}{l} < 0$$

(i.e. we indeed have two imaginary frequencies) while the generalised eigenvectors  $\mathbf{u} \propto \begin{pmatrix} 1 \\ \mp \sqrt{2} \end{pmatrix}$  remain the same.



# Chapter 4

## Rigid bodies

An important application of mechanics is the motion of rigid bodies. Rigid bodies are formally defined as follows:

A **rigid body** is a system in which the distances between the particles do not vary in time.

Most solid objects around us are rigid bodies to a reasonable approximation, because the distances between the atoms do not vary in time. The distances of the particles being fixed, there are only few things that one can do with a rigid body: **translate** it (i.e. move all particles by the same amount in the same direction) and **rotate** it about an arbitrary axis.

In this lecture we will only be interested in **rotations about axes through the origin**. These rotations are important in the following two cases:

- If the body is **fixed at the origin**, we can no longer move all particles by the same amount in the same direction. And if we want to rotate the body we must make sure that the origin remains fixed. This means that the axis of rotation must go through the origin.
- Now let us consider a rigid body on which **no outside forces** are acting, e.g. a rigid body in space. In this case it is helpful to study the motion of the centre of mass, defined by

$$\sum_{i=1}^N \frac{m_i}{M} \mathbf{r}_i$$

where  $M = \sum_{i=1}^N m_i$  is the overall mass of the system. As any isolated system, such a rigid body satisfies several conservation laws (see Section 2.4.4). In particular the overall momentum of all particles is conserved,

$$\sum_{i=1}^N m_i \dot{\mathbf{r}}_i = \text{const} .$$

This implies that

$$\frac{d}{dt} \sum_{i=1}^N \frac{m_i}{M} \mathbf{r}_i = \text{const} ,$$

i.e., the velocity of the centre of mass is constant.

We now consider the case of zero velocity, and choose our coordinate system such that the **centre of mass is at the origin**. Then again only rotations about axes through the origin are possible.

Note that the body is still allowed to rotate about **all** axes through the origin, and that the rotation axis can change in time. A main goal will be to understand such changes of the rotation axis.

## 4.1 Angular velocity

The angular velocity is an important quantity that characterises both the axis and the speed of rotation.

**Def.:** Assume that a rigid body is rotated about an axis that goes through the origin and has the direction  $\mathbf{n}$  (where  $\mathbf{n}$  is a unit vector). If during the time  $dt$  the body is rotated by an angle  $d\phi$ , its **angular velocity** is defined as

$$\boldsymbol{\omega} = \frac{d\phi}{dt} \mathbf{n}.$$

If we know the angular velocity we can determine the velocity of every particle in the body.

**Fact:** If a rigid body rotates with an angular velocity  $\boldsymbol{\omega}$ , the velocities of the particles are given by

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i.$$

**Proof:**

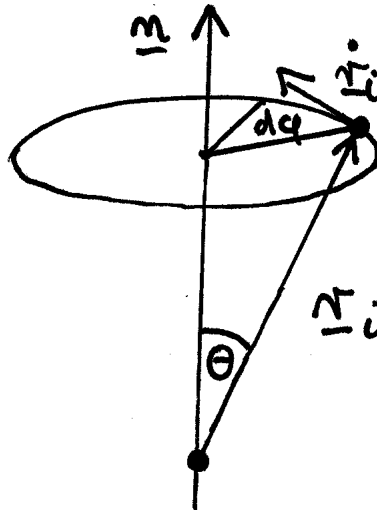


Figure 4.1: Rotation about an axis through the origin with direction  $\mathbf{n}$ .

We show that both the norms and the directions of  $\dot{\mathbf{r}}_i$  and  $\boldsymbol{\omega} \times \mathbf{r}_i$  coincide. We use that a particle rotating about an axis moves on a circle. As seen in the picture the radius of this circle is  $|\mathbf{r}_i| \sin \theta$  where  $|\mathbf{r}_i|$  is the distance from the origin and  $\theta$

is the angle enclosed between the position vector and the direction of the axis. The product  $|\mathbf{r}_i| \sin \theta$  can also be written as  $|\mathbf{n} \times \mathbf{r}_i|$  (using that  $\mathbf{n}$  is normalised). A particle moved by an angle  $d\phi$  now travels a distance  $|\mathbf{n} \times \mathbf{r}_i|d\phi$  on the circle. The absolute value of its velocity is obtained by dividing out  $dt$ . We then get

$$|\dot{\mathbf{r}}_i| = \left| \underbrace{\mathbf{n} \frac{d\phi}{dt}}_{=\boldsymbol{\omega}} \times \mathbf{r}_i \right|, \quad (4.1)$$

as desired.

Now we consider the direction of the vectors. The circle that our particle moves on is in a plane perpendicular to the rotation axis, which means that the velocity is perpendicular to  $\mathbf{n}$ . In addition it is clear from the picture that the velocity is perpendicular to  $\mathbf{r}_i$ . The same holds true for the cross product  $\mathbf{n} \times \mathbf{r}_i$ . Hence also the directions coincide and the statement is proven.

## 4.2 Inertia tensor

### Analogies

It would be nice if we could describe the rotation of rigid bodies in a way analogous to to a motion that we already know very well: the translational motion of a single particle. The most important properties of such a particle are its momentum  $\mathbf{p}$ , its velocity  $\mathbf{v}$  and its mass  $m$ . These quantities are related by  $\mathbf{p} = m\mathbf{v}$ . For a rigid body, the analogue of the momentum is the total angular momentum, and the analogue of the velocity is the angular velocity. The question is therefore: Is there an analogue of the mass as well, and can we use this to write an equation similar to  $\mathbf{p} = m\mathbf{v}$ ?

We will see that an analogue of the mass indeed exists; however it is not a really number but a matrix – the so-called **inertia tensor**. To find it we write the total angular momentum as

$$\begin{aligned} \mathbf{l} &= \sum_{i=1}^N \mathbf{r}_i \times \mathbf{p}_i \\ &= \sum_{i=1}^N \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i \\ &= \sum_{i=1}^N \mathbf{r}_i \times m_i (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &= \sum_{i=1}^N m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i). \end{aligned} \quad (4.2)$$

Reassuringly, Eq. (4.2) already expresses the total angular momentum as a linear function of the angular velocity. To bring it closer to the form  $\mathbf{p} = m\mathbf{v}$  we use the following formula for combinations of two cross products:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

(the so-called “back-cab formula”). We then obtain

$$\mathbf{l} = \sum_{i=1}^N m_i \{ \boldsymbol{\omega}(\mathbf{r}_i \cdot \mathbf{r}_i) - \mathbf{r}_i(\mathbf{r}_i \cdot \boldsymbol{\omega}) \} .$$

It is now helpful to write out the  $a$ th component of the total angular momentum. Since the  $\mathbf{r}_i \cdot \mathbf{r}_i$  in the first term and the  $\mathbf{r}_i \cdot \boldsymbol{\omega}$  in the second term are just real numbers, for the  $a$ th component we have to replace the  $\boldsymbol{\omega}$  in the first term by  $\omega_a$  and the  $\mathbf{r}_i$  in the second term by  $r_{ia}$ . Writing out the scalar products we thus obtain

$$l_a = \sum_{i=1}^N m_i \left\{ \omega_a \sum_{c=1}^3 r_{ic}^2 - r_{ia} \sum_{b=1}^3 r_{ib} \omega_b \right\}$$

To let the formula look more symmetric, it would be helpful to introduce a summation over  $b$  not only in the second term, but also in the first one. We thus write

$$\omega_a = \sum_{b=1}^3 \delta_{ab} \omega_b$$

with the Kronecker delta already used in Section 2.4.1. If we now pull the sum over  $b$  in front of all other terms and shift  $\omega_b$  to the end, we can write the  $a$ th component of the angular momentum as

$$l_a = \sum_{b=1}^3 \sum_{i=1}^N m_i \left\{ \delta_{ab} \sum_{c=1}^3 r_{ic}^2 - r_{ia} r_{ib} \right\} \omega_b .$$

We thus obtain the following result:

The total angular momentum  $\mathbf{l}$  and the angular velocity  $\boldsymbol{\omega}$  are related by

$$l_a = \sum_{b=1}^3 I_{ab} \omega_b$$

where

$$I_{ab} = \sum_{i=1}^N m_i \left\{ \delta_{ab} \sum_{c=1}^3 r_{ic}^2 - r_{ia} r_{ib} \right\} .$$

In matrix notation we have

$$\mathbf{l} = I \boldsymbol{\omega}$$

where

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

is called the **inertia tensor**.

### Rotating coordinate system

The inertia tensor  $I$  defined above depends on the coordinates  $r_{ia}$  which themselves are functions of time. This can be avoided if we write our vectors in a basis that depends on time:

In the following, we will use basis vectors  $\mathbf{E}_1(t)$ ,  $\mathbf{E}_2(t)$ ,  $\mathbf{E}_3(t)$  that depend on time and rotate in the same way as the particle positions in the rigid body. This basis is chosen to be orthonormal, i.e., we have

$$\mathbf{E}_A(t) \cdot \mathbf{E}_B(t) = \delta_{AB} . \quad (4.3)$$

Moreover we want to have a “right-handed” basis, which means that

$$\mathbf{E}_1(t) \times \mathbf{E}_2(t) = \mathbf{E}_3(t), \quad \mathbf{E}_2(t) \times \mathbf{E}_3(t) = \mathbf{E}_1(t), \quad \mathbf{E}_3(t) \times \mathbf{E}_1(t) = \mathbf{E}_2(t) . \quad (4.4)$$

**Example:** If the body rotates about the  $z$ -axis with angular velocity  $\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$  we can use

$$\mathbf{E}_1(t) = \begin{pmatrix} \cos \omega t \\ \sin \omega t \\ 0 \end{pmatrix}, \quad \mathbf{E}_2(t) = \begin{pmatrix} -\sin \omega t \\ \cos \omega t \\ 0 \end{pmatrix}, \quad \mathbf{E}_3(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

**Expansion in the new basis:** All vectors  $\mathbf{v}(t)$  can now be expanded in the basis  $\mathbf{E}_1(t)$ ,  $\mathbf{E}_2(t)$ ,  $\mathbf{E}_3(t)$  and the coefficients will be denoted by capital letters,

$$\mathbf{v}(t) = \sum_{A=1}^3 V_A(t) \mathbf{E}_A(t)$$

For instance we will have

$$\begin{aligned} \mathbf{r}_i(t) &= \sum_{A=1}^3 R_{iA} \mathbf{E}_A(t) \\ \boldsymbol{\omega}(t) &= \sum_{A=1}^3 \Omega_A(t) \mathbf{E}_A(t) \\ \mathbf{l}(t) &= \sum_{A=1}^3 L_A(t) \mathbf{E}_A(t) . \end{aligned}$$

Note that here the new particle coordinates  $R_{iA}$  don't depend on time, because the basis vectors are rotated in the same way as the particles and the whole time dependence of  $\mathbf{r}_i(t)$  is now in the rotation of the basis vectors  $\mathbf{E}_A(t)$ . That's the reason why we introduced these basis vectors in the first place. We don't know yet how the total angular momentum and the angular velocity are going to depend on time, so in contrast to  $R_{iA}$  the new coordinates of these quantities still have to be written as functions of  $t$ .

Due to Eq. (4.3) all scalar products of vectors can be written as sums over products of their components in the new basis:

$$\begin{aligned} \mathbf{v}(t) \cdot \mathbf{w}(t) &= \sum_A V_A(t) \mathbf{E}_A(t) \cdot \sum_B W_B(t) \mathbf{E}_B(t) \\ &= \sum_A \sum_B V_A(t) W_B(t) \delta_{AB} \\ &= \sum_A V_A(t) W_A(t) . \end{aligned}$$

For scalar products with basis vectors this implies

$$\mathbf{v}(t) \cdot \mathbf{E}_A(t) = V_A(t) .$$

To apply these ideas to Eq. (4.2) we multiply with  $\mathbf{E}_A(t)$ . Repeated application of the formula for the scalar product and manipulations similar to the previous calculation give

$$\begin{aligned} L_A(t) &= \mathbf{l}(t) \cdot \mathbf{E}_A(t) \\ &= \sum_{i=1}^N m_i [(\boldsymbol{\omega}(t) \cdot \mathbf{E}_A(t))(\mathbf{r}_i(t) \cdot \mathbf{r}_i(t)) - (\mathbf{r}_i(t) \cdot \mathbf{E}_A(t))(\mathbf{r}_i(t) \cdot \boldsymbol{\omega}(t))] \\ &= \sum_{i=1}^N m_i \left[ \underbrace{\Omega_A(t)}_{=\sum_B \delta_{AB} \Omega_B(t)} \sum_C R_{iC}^2 - R_{iA} \sum_B R_{iB} \Omega_B \right] \\ &= \sum_B \left[ \underbrace{\sum_{i=1}^N m_i \left( \delta_{AB} \sum_C R_{iC}^2 - R_{iA} R_{iB} \right)}_{\equiv \mathcal{I}_{AB}} \right] \Omega_B(t) \\ &= \sum_B \mathcal{I}_{AB} \Omega_B(t) . \end{aligned}$$

Here  $\mathcal{I}_{AB}$  is independent of time, like the mass in  $\mathbf{p} = m\dot{\mathbf{r}}$ ! In many cases  $\mathcal{I}_{AB}$  can be made diagonal by choosing a convenient set of rotating vectors. The diagonal elements obtained in this way (i.e. the eigenvalues of  $\mathcal{I}$  are also called **moments of inertia**).

### Continuously distributed mass

Typically a rigid body contains lots of particles (atoms). In this case a further improvement is very helpful. Instead of summing over all positions  $\mathbf{R}_i$  of atoms, we assume that the mass is distributed continuously over the rigid body. It is helpful to describe this distribution in terms of the mass density:

**Def.:** The **mass density**  $\rho(R_1, R_2, R_3)$  of a rigid body is defined such that the mass in a volume from  $R_1$  to  $R_1 + dR_1$ , from  $R_2$  to  $R_2 + dR_2$ , and from  $R_3$  to  $R_3 + dR_3$  is given by  $\rho(R_1, R_2, R_3)dR_1dR_2dR_3$ .

Typically, the mass is just distributed uniformly over the rigid body. Then the mass density is simply the overall mass of the body divided by the overall volume.

The elements of the inertia tensor can now be determined as by

$$\mathcal{I}_{AB} = \int \int \int \rho(R_1, R_2, R_3) \left( \delta_{AB} \sum_C R_C^2 - R_A R_B \right) dR_1 dR_2 dR_3 \quad (4.5)$$

**Proof:** The contribution of the volume from  $R_1$  to  $R_1 + dR_1$ , from  $R_2$  to  $R_2 + dR_2$ , and from  $R_3$  to  $R_3 + dR_3$  to  $\mathcal{I}_{AB}$  is given by

$$\rho(R_1, R_2, R_3)dR_1dR_2dR_3 \left( \delta_{AB} \sum_C R_C^2 - R_A R_B \right) .$$

Integration over the whole body gives Eq. (4.5).



**Example**

An example for a rigid body is a box with coordinates  $-\frac{a}{2} < R_1 < \frac{a}{2}$ ,  $-\frac{b}{2} < R_2 < \frac{b}{2}$  and  $-\frac{c}{2} < R_3 < \frac{c}{2}$  and a constant density. If we denote the mass of the box by  $M$  and use that the volume is  $abc$  we thus have  $\rho = \frac{M}{abc} = \text{const}$ . Application of Eq. (4.5) then gives

$$\begin{aligned} \mathcal{I} &= \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \rho \begin{pmatrix} R_2^2 + R_3^2 & -R_1 R_2 & -R_1 R_3 \\ -R_2 R_1 & R_1^2 + R_3^2 & -R_2 R_3 \\ -R_3 R_1 & -R_3 R_2 & R_1^2 + R_2^2 \end{pmatrix} dR_1 dR_2 dR_3 \\ &= \underbrace{\rho abc}_{=M} \begin{pmatrix} \frac{b^2+c^2}{12} & 0 & 0 \\ 0 & \frac{a^2+c^2}{12} & 0 \\ 0 & 0 & \frac{a^2+b^2}{12} \end{pmatrix} \end{aligned}$$

where different rows and columns now correspond to different directions in the rotating coordinate system fixed to the rigid body. The off-diagonal elements vanish due to  $\int_{-a/2}^{a/2} R_1 dR_1 = 0$ , etc. We see that  $\mathcal{I}$  is constant in time and even diagonal.

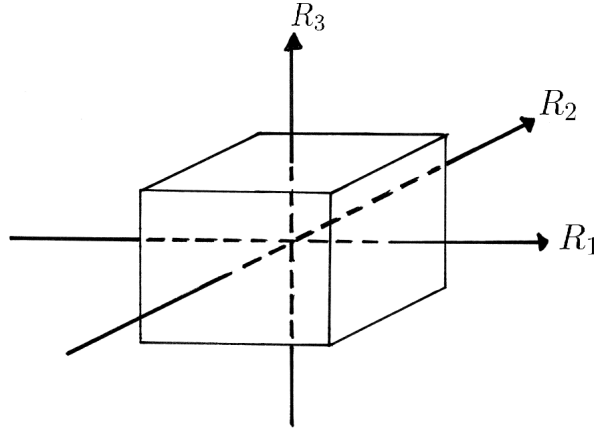


Figure 4.2: A box centred at the origin.

**Example (optional)**

Consider a sphere with radius  $R$ , volume  $V = \frac{4}{3}\pi R^3$  and constant density  $\rho = \frac{M}{V}$ . We use spherical coordinates  $R_1 = r \sin \theta \cos \phi$ ,  $R_2 = r \sin \theta \sin \phi$ ,  $R_3 = r \cos \theta$ . The inertia tensor is

$$\begin{aligned} \mathcal{I} &= \int_0^R \int_0^\pi \int_0^{2\pi} \rho \begin{pmatrix} R_2^2 + R_3^2 & -R_1 R_2 & -R_1 R_3 \\ -R_2 R_1 & R_1^2 + R_3^2 & -R_2 R_3 \\ -R_3 R_1 & -R_3 R_2 & R_1^2 + R_2^2 \end{pmatrix} r^2 \sin \theta d\phi d\theta dr \\ &= \int_0^R \int_0^\pi \int_0^{2\pi} \rho r^2 \begin{pmatrix} \sin^2 \theta \sin^2 \phi + \cos^2 \theta & -\sin^2 \theta \cos \phi \sin \phi & -\sin \theta \cos \phi \cos \theta \\ -\sin^2 \theta \sin \phi \cos \phi & \sin^2 \theta \cos^2 \phi + \cos^2 \theta & -\sin \theta \sin \phi \cos \theta \\ -\cos \theta \sin \theta \cos \phi & -\cos \theta \sin \theta \sin \phi & \sin^2 \theta \end{pmatrix} r^2 \sin \theta d\phi d\theta dr \\ &= \underbrace{\frac{M}{\frac{4}{3}\pi R^3}}_{=\rho} \cdot \frac{1}{5} R^5 \cdot 2\pi \cdot \frac{4}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{2}{5} M R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Here the off-diagonal elements vanish due to  $\int_0^{2\pi} \cos \phi \sin \phi d\phi = \int_0^{2\pi} \frac{1}{2} \sin 2\phi = 0$ ,  $\int_0^{2\pi} \cos \phi d\phi = 0$ ,  $\int_0^{2\pi} \sin \phi d\phi = 0$ . For the diagonal elements we need the  $\phi$ -integrals  $\int_0^{2\pi} \cos^2 \phi d\phi = \int_0^{2\pi} \sin^2 \phi d\phi = \pi$ , the  $\theta$ -integrals  $\int_0^\pi \cos^2 \theta \sin \theta = -\frac{1}{3} \cos^2 \theta \Big|_0^\pi = \frac{2}{3}$ ,  $\int_0^\pi \sin^3 \theta d\theta = \int_0^\pi \sin \theta d\theta - \int_0^\pi \cos^2 \theta \sin \theta = 2 - \frac{2}{3} = \frac{4}{3}$ , and the  $r$ -integral  $\int_0^R r^4 dr = \frac{1}{5} R^5$ .

### 4.3 Euler's equations

We now want to find out how for a rigid body the angular velocity components  $\Omega_1(t), \Omega_2(t), \Omega_3(t)$  (and thus the axis and speed of rotation) change in time. For simplicity we only consider rigid bodies on which no external forces are acting. Then we use that for such isolated systems the total angular momentum  $\mathbf{l}$  is a conserved quantity. We thus obtain

$$\begin{aligned}
 0 &= \frac{d}{dt} \mathbf{l} \\
 &= \frac{d}{dt} \sum_A L_A(t) \mathbf{E}_A(t) \\
 &= \dot{L}_1(t) \mathbf{E}_1(t) + \dot{L}_2(t) \mathbf{E}_2(t) + \dot{L}_3(t) \mathbf{E}_3(t) + L_1(t) \dot{\mathbf{E}}_1(t) + L_2(t) \dot{\mathbf{E}}_2(t) + L_3(t) \dot{\mathbf{E}}_3(t) .
 \end{aligned} \tag{4.6}$$

(Note that the conservation of  $\mathbf{l}$  does not imply conservation of  $L_1(t)$ ,  $L_2(t)$  and  $L_3(t)$ !) We now need the derivatives of the basis vectors  $\mathbf{E}_1(t)$ ,  $\mathbf{E}_2(t)$ ,  $\mathbf{E}_3(t)$ . Since these basis vectors are rotated in the same way as the particle positions their time derivatives are also obtained by taking a cross product with  $\boldsymbol{\omega}(t)$ . This gives

$$\begin{aligned}
 \dot{\mathbf{E}}_1(t) &= \boldsymbol{\omega}(t) \times \mathbf{E}_1(t) \\
 &= (\Omega_1 \mathbf{E}_1(t) + \Omega_2(t) \mathbf{E}_2(t) + \Omega_3(t) \mathbf{E}_3(t)) \times \mathbf{E}_1(t) \\
 &= -\Omega_2(t) \mathbf{E}_3(t) + \Omega_3(t) \mathbf{E}_2(t)
 \end{aligned}$$

where in the second line  $\boldsymbol{\omega}(t)$  was expanded in terms of the rotating basis vectors and in the third line I used that we have a right-handed coordinate system (Eq. (4.4)) and that the cross product  $\mathbf{E}_1(t) \times \mathbf{E}_1(t)$  is zero. Analogous reasoning for the other basis vectors gives

$$\begin{aligned}
 \dot{\mathbf{E}}_2(t) &= \Omega_1(t) \mathbf{E}_3(t) - \Omega_3(t) \mathbf{E}_1(t) \\
 \dot{\mathbf{E}}_3(t) &= -\Omega_1(t) \mathbf{E}_2(t) + \Omega_2(t) \mathbf{E}_1(t) .
 \end{aligned}$$

If we substitute all this into Eq. (4.6) and collect the terms proportional to each of the basis vectors we get

$$\begin{aligned}
 \dot{L}_1(t) + \Omega_2(t) L_3(t) - \Omega_3(t) L_2(t) &= 0 \\
 \dot{L}_2(t) + \Omega_3(t) L_1(t) - \Omega_1(t) L_3(t) &= 0 \\
 \dot{L}_3(t) + \Omega_1(t) L_2(t) - \Omega_2(t) L_1(t) &= 0 .
 \end{aligned} \tag{4.7}$$

Now we can eliminate the  $L_A(t)$ 's using that  $L_A(t) = \sum_B \mathcal{I}_{AB} \Omega_B(t)$ . If we choose our rotating basis in such a way that  $\mathcal{I}$  becomes diagonal (with diagonal elements  $\mathcal{I}_1, \mathcal{I}_2$  and  $\mathcal{I}_3$ ), this relation turns into

$$L_A(t) = \mathcal{I}_A \Omega_A(t) .$$

This implies

$$\dot{L}_A(t) = \mathcal{I}_A \dot{\Omega}_A(t) .$$

Eq. (4.7) thus turns into

**Euler's equations**

$$\begin{aligned}
 \mathcal{I}_1 \dot{\Omega}_1 &= (\mathcal{I}_2 - \mathcal{I}_3) \Omega_2 \Omega_3 \\
 \mathcal{I}_2 \dot{\Omega}_2 &= (\mathcal{I}_3 - \mathcal{I}_1) \Omega_3 \Omega_1 \\
 \mathcal{I}_3 \dot{\Omega}_3 &= (\mathcal{I}_1 - \mathcal{I}_2) \Omega_1 \Omega_2
 \end{aligned} \tag{4.8}$$

where the  $\Omega_A$ 's depend on time whereas the  $\mathcal{I}$ 's do not.

**Note:** Alternatively we could have used  $\Omega_A = \frac{L_A}{\mathcal{I}_A}$  to eliminate the  $\Omega_A$ 's, yielding

$$\begin{aligned}
 \dot{L}_1 &= \frac{\mathcal{I}_2 - \mathcal{I}_3}{\mathcal{I}_2 \mathcal{I}_3} L_2 L_3 \\
 \dot{L}_2 &= \frac{\mathcal{I}_3 - \mathcal{I}_1}{\mathcal{I}_3 \mathcal{I}_1} L_3 L_2 \\
 \dot{L}_3 &= \frac{\mathcal{I}_1 - \mathcal{I}_2}{\mathcal{I}_1 \mathcal{I}_2} L_1 L_3 .
 \end{aligned}$$

Euler's equations (4.8) determine the time evolution of the  $\Omega_A$ 's and thus of the rotation axis. Unfortunately, they are nonlinear (the  $\Omega_A$ 's and their derivatives appear both linearly and quadratically) and thus difficult to solve. However, there is a special case (the symmetric top) in which they effectively become linear and can be solved easily.

**Example: Symmetric top**

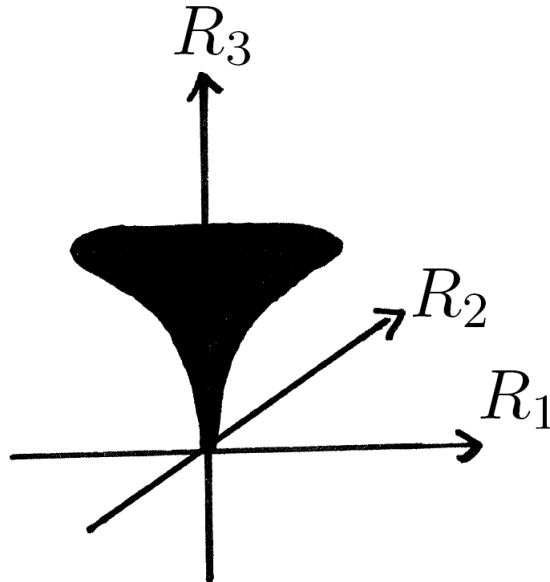


Figure 4.3: The symmetric top.

An example for a symmetric top is shown in Fig. 4.3. We see that the top is symmetric w.r.t. rotations around the  $R_3$ -axis. Hence the moment of inertia  $\mathcal{I}_1$

corresponding to the  $R_1$ -axis and the moment of inertia  $\mathcal{I}_2$  corresponding to the  $R_2$ -axis should coincide, but differ from the moment of inertia  $\mathcal{I}_3$  corresponding to rotations around the  $R_3$ -axis. Denoting the two different moments of inertia by  $\mathcal{I}_\perp$  and  $\mathcal{I}_\parallel$ , we thus have

$$\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_\perp \neq \mathcal{I}_3 = \mathcal{I}_\parallel .$$

If we insert these momenta into Euler's equations (4.8), we find

$$\begin{aligned} \mathcal{I}_\perp \dot{\Omega}_1 &= (\mathcal{I}_\perp - \mathcal{I}_\parallel) \Omega_2 \Omega_3 \\ \mathcal{I}_\perp \dot{\Omega}_2 &= (\mathcal{I}_\parallel - \mathcal{I}_\perp) \Omega_3 \Omega_1 \\ \mathcal{I}_\parallel \dot{\Omega}_3 &= 0 \end{aligned} \tag{4.9}$$

Crucially, the third line implies that the time derivative of  $\Omega_3$  vanishes. Hence  $\Omega_3$  (the component of the angular velocity corresponding to rotations around the  $Z$ -axis) is a constant. The only thing we still have to do is to solve for  $\Omega_1$  and  $\Omega_2$ . – This is now a simple task since the first two equations are linear in  $\Omega_1$  and  $\Omega_2$ . To proceed we collect the constants  $\mathcal{I}_\parallel$ ,  $\mathcal{I}_\perp$  and  $\Omega_3$  into one, denoted by

$$\nu = \frac{\mathcal{I}_\perp - \mathcal{I}_\parallel}{\mathcal{I}_\perp} \Omega_3 = \text{const.}$$

The first two equations in (4.9) thus turn into

$$\dot{\Omega}_1 = -\nu \Omega_2 \tag{4.10}$$

$$\dot{\Omega}_2 = \nu \Omega_1 \tag{4.11}$$

These equations can now easily be solved. If we take the derivative of (4.10) and insert (4.11) we get

$$\ddot{\Omega}_1 = -\nu \dot{\Omega}_2 = -\nu^2 \Omega_1 .$$

This is the well-known equation of a harmonic oscillator. The solution reads

$$\Omega_1 = C \cos(\nu t + \phi) ,$$

where  $C$  and  $\phi$  are constants that have to be chosen to satisfy the initial conditions.  $\Omega_2$  can now be determined from Eq. (4.10) as

$$\Omega_2 = -\frac{1}{\nu} \dot{\Omega}_1 = C \sin(\nu t + \phi)$$

These equations just indicate a rotation about the  $R_3$ -axis with frequency  $\nu$ . We thus see that the rotation axis of a symmetric top is not fixed (even in a coordinate system following the rigid body), but precesses about the  $R_3$ -axis.

### Example: Rotation axis of the Earth

Another example for a rigid body is the Earth. Of course, the Earth has an almost spherical shape. But it is not quite a sphere – the radius of the Equator is a little bit larger than the distance between the Equator plane and the North pole. Hence, just like for the symmetric top the moments of inertia corresponding to the  $R_1$ - and  $R_2$ -direction coincide with each other, but differ slightly from the moment of inertia

corresponding to the  $R_3$ -axis. Therefore, Leonhard Euler courageously applied to the Earth the same results as for the symmetric top. For the Earth we have

$$\frac{\mathcal{I}_{\parallel} - \mathcal{I}_{\perp}}{\mathcal{I}_{\perp}} \approx \frac{1}{305}$$

and (since it rotates around itself once in a day)

$$\Omega_3 = \frac{2\pi}{1 \text{ day}} .$$

One would thus conclude that the Earth's rotation axis rotates (precedes) relative to the surface of the Earth with a frequency

$$\nu = \frac{2\pi}{305 \text{ days}} ,$$

i.e., with a period of 305 days. Measurements by Set Carlo Chandler (1846-1913) confirmed the existence of this precession (which is now also called the **Chandler wobble**), but he got a period of 433 days! This discrepancy was explained by Simon Newcomb (1835-1909) as being due to the slight non-rigidity of the Earth. Actually the Earth is quite rigid, slightly more rigid than steel – but the difference from a perfect rigid body is comparable to the difference of the shape of the Earth from a perfect sphere, which was causing the effect in the first place. It is not possible to keep into account one of these small effects and neglect the other.



# Chapter 5

## Hamiltonian mechanics

### 5.1 Hamilton's equations

Hamiltonian mechanics is a formulation of mechanics similar to the one by Lagrange, but with important advantages. To motivate it, let us recall some results from the previous chapters.

#### Reminder

We started from **Newtonian mechanics**, in particular from Newton's second law  $\mathbf{F} = m\mathbf{a}$ . We then derived **Lagrange's** formulation of mechanics. The main advantage of Lagrange's formalism is that it looks the same for all coordinate systems, and one can easily account for constraints. The central object in Lagrangian mechanics is the **Lagrangian**  $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T - U$ , depending on the generalised coordinates  $\mathbf{q}$ , their derivatives  $\dot{\mathbf{q}}$  and time  $t$ . The trajectories of particles are then obtained from Lagrange's equations. To solve Lagrange's equations it is very helpful to identify **conserved quantities** of the system. These conserved quantities can be read off from the Lagrangian:

- If the Lagrangian is independent of time the **generalised energy**

$$h(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

is conserved.

- If the Lagrangian is independent of one of the coordinates  $q_\alpha$ , the corresponding **generalised momentum**

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \tag{5.1}$$

is conserved.

#### Hamilton

Even though conservation laws can be used in Lagrangian mechanics, the fundamental quantities in Lagrangian mechanics are still the Lagrangian  $L$ , the coordinates  $\mathbf{q}$  and their derivatives  $\dot{\mathbf{q}}$ , not the (potentially) conserved quantities  $h$  and  $\mathbf{p}$ . The treatment of conservation laws would be much cleaner if instead we could **build**

**the whole theory around  $h$  and  $\mathbf{p}$ .** We would thus replace the Lagrangian as the main quantity of interest by the generalised energy, and use the momenta  $\mathbf{p}$  instead of the derivatives  $\dot{\mathbf{q}}$ . This is the idea of Hamiltonian mechanics.

To replace  $\dot{\mathbf{q}}$  one can use that  $\mathbf{p}$  is determined by  $\mathbf{q}$ ,  $\dot{\mathbf{q}}$  and  $t$  through (5.1)

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \mathbf{p}(\mathbf{q}, \dot{\mathbf{q}}, t) . \quad (5.2)$$

We now assume that this equation can be solved uniquely for  $\dot{\mathbf{q}}$ . – This uniqueness is a condition for the applicability of Hamiltonian mechanics. Then  $\dot{\mathbf{q}}$  can be written as a function of  $\mathbf{q}$ ,  $\mathbf{p}$  and  $t$ ,

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t) . \quad (5.3)$$

We now want to build a theory based on the generalised energy  $h$ . But instead of writing  $h$  as a function of  $\mathbf{q}$ ,  $\dot{\mathbf{q}}$  and  $t$  we want to replace  $\dot{\mathbf{q}}$  as an argument by the momentum  $\mathbf{p}$ . This can be done if we substitute all occurrences of  $\dot{\mathbf{q}}$  by  $\dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)$  and replace  $\frac{\partial L}{\partial \dot{\mathbf{q}}}$  by  $\mathbf{p}$ . The resulting function is called the Hamiltonian.

### Hamiltonian

$$H(\mathbf{q}, \mathbf{p}, t) = h(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t), t) = \mathbf{p} \cdot \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t) - L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t), t)$$

The Hamiltonian can be used to get equations of motion similar to Lagrange's equations. These equations have the following form:

### Hamilton's equations

$$\begin{aligned} \dot{q}_\alpha &= \frac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha &= -\frac{\partial H}{\partial q_\alpha} \end{aligned} \quad (5.4)$$

**Proof:** To prove Hamilton's equations we compute the partial derivatives of the Hamiltonian. There are three terms in  $H$  that depend on  $H$ : the initial  $\mathbf{p}$  and two functions  $\dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)$ . Taking into account all these terms we obtain the derivative  $\frac{\partial H}{\partial p_\alpha}$  as

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha + \mathbf{p} \cdot \frac{\partial \dot{\mathbf{q}}}{\partial p_\alpha} - \underbrace{\frac{\partial L}{\partial \dot{\mathbf{q}}}}_{=\mathbf{p}} \cdot \frac{\partial \dot{\mathbf{q}}}{\partial p_\alpha} = \dot{q}_\alpha$$

Here the second and third term cancel due to  $\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}$ . The first equation in (5.4) is thus proven.

To compute  $\frac{\partial H}{\partial q_\alpha}$  we take into account the  $\mathbf{q}$ -dependence of the two  $\dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)$ 's and of the explicit dependence of  $L$  on  $\mathbf{q}$ . We thus get

$$\frac{\partial H}{\partial q_\alpha} = \mathbf{p} \cdot \frac{\partial \dot{\mathbf{q}}}{\partial q_\alpha} - \frac{\partial L}{\partial q_\alpha} - \underbrace{\frac{\partial L}{\partial \dot{\mathbf{q}}}}_{=\mathbf{p}} \cdot \frac{\partial \dot{\mathbf{q}}}{\partial q_\alpha} = -\frac{\partial L}{\partial q_\alpha}$$



where again two terms have cancelled. If we now use Lagrange's equation  $\frac{\partial L}{\partial q_\alpha} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha}$  and the definition of the momentum we can write the result as

$$\frac{\partial H}{\partial q_\alpha} = - \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{q}_\alpha}}_{=p_\alpha} = -\dot{p}_\alpha ,$$

proving second equation of (5.4).

A related result concerns the **time derivative** of the Hamiltonian:

$$\frac{\partial H}{\partial t} = - \frac{\partial L}{\partial t}$$

**Proof:** Differentiating w.r.t. all three occurrences of  $t$  and again using the definition of  $\mathbf{p}$  we get

$$\frac{\partial H}{\partial t} = \mathbf{p} \cdot \frac{\partial \dot{\mathbf{q}}}{\partial t} - \underbrace{\frac{\partial L}{\partial \dot{\mathbf{q}}}}_{=\mathbf{p}} \cdot \frac{\partial \dot{\mathbf{q}}}{\partial t} - \frac{\partial L}{\partial t} = - \frac{\partial L}{\partial t}$$

**Examples:**

**a) Particle in 1d**

A particle of mass  $m$  in moving in one dimension has the Lagrangian

$$L = \frac{1}{2} m \dot{q}^2 - U(q)$$

where  $U(q)$  is the potential. The generalised **momentum** is obtained as

$$p = \frac{\partial L}{\partial \dot{q}} = m \dot{q} , \quad (5.5)$$

i.e., it is simply the linear momentum (mass times velocity). Eq. (5.5) can be solved for  $\dot{q}$ ,

$$\dot{q} = \frac{p}{m} . \quad (5.6)$$

The **Hamiltonian** is now obtained as

$$H(q, p, t) = p \dot{q} - L$$

where all  $\dot{q}$ 's have to be replaced by  $\frac{p}{m}$ . We thus get

$$H(q, p, t) = \frac{p^2}{m} - \left( \frac{p^2}{2m} - U(q) \right) = \frac{p^2}{2m} + U(q) ,$$

and **Hamilton's equations** read

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad (5.7)$$

$$\dot{p} = - \frac{\partial H}{\partial q} = - \frac{\partial U}{\partial q} . \quad (5.8)$$

Here (5.7) is equivalent to the definition of  $p$  (see Eq. (5.6)); hence this equation merely provides a consistency check, whereas (5.8) gives something new. We also note that Eqs. (5.7) and (5.8) could be combined into

$$\ddot{q} = \frac{\dot{p}}{m} = -\frac{1}{m} \frac{\partial U}{\partial q}. \quad (5.9)$$

Since  $-\frac{\partial U}{\partial q}$  is the conservative force corresponding to the potential  $U(q)$ , Eq. (5.9) coincides with Newton's second law.

### b) Particle in 2d, polar coordinates

The Lagrangian of a particle in two dimensions can be written in polar coordinates  $\rho, \phi$  as

$$L = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 - U(\rho, \phi).$$

We can now determine the **momenta** associated to  $\rho$  and  $\phi$  and solve for  $\dot{\rho}$  and  $\dot{\phi}$ ,

$$\begin{aligned} p_\rho &= \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} \implies \dot{\rho} = \frac{p_\rho}{m} \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} \implies \dot{\phi} = \frac{p_\phi}{m\rho^2}. \end{aligned}$$

The **Hamiltonian** is thus given by

$$\begin{aligned} H(\rho, \phi, p_\rho, p_\phi, t) &= p_\rho\dot{\rho} + p_\phi\dot{\phi} - L \\ &= \frac{p_\rho^2}{m} + \frac{p_\phi^2}{m\rho^2} - \left( \frac{p_\rho^2}{2m} + \frac{p_\phi^2}{2m\rho^2} - U(\rho, \phi) \right) \\ &= \frac{p_\rho^2}{2m} + \frac{p_\phi^2}{2m\rho^2} + U(\rho, \phi), \end{aligned}$$

and **Hamilton's equations** read

$$\begin{aligned} \dot{\rho} &= \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} \\ \dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2} \\ \dot{p}_\rho &= -\frac{\partial H}{\partial \rho} = \frac{p_\phi^2}{m\rho^3} - \frac{\partial U}{\partial \rho} \\ \dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = -\frac{\partial U}{\partial \phi}. \end{aligned}$$

Here the first two equations again provide a consistency check with the definitions of  $p_\rho$  and  $p_\phi$ . The third and fourth equation give the time derivative of the momenta. Again these time derivatives involve partial derivatives of the potential.

Finally we mention that for a central field (a potential  $U$  that depends only on  $\rho$  and thus satisfies  $\frac{\partial U}{\partial \phi} = 0$ ) the angular momentum  $p_\phi$  is a conserved quantity (i.e.  $\dot{p}_\phi = 0$ ). In the context of Lagrangian mechanics this was already seen in Section 2.4.2.

## c) Quadratic kinetic energy

In the two previous examples the Hamiltonian coincided with the energy  $E = T + U$ , written as a function of  $\mathbf{q}$ ,  $\mathbf{p}$  and  $t$ . This agrees with our previous observation for the generalised energy  $h$ , the only difference being that the Hamiltonian is written as a function of  $\mathbf{q}$ ,  $\mathbf{p}$  and  $t$  rather than  $\mathbf{q}$ ,  $\dot{\mathbf{q}}$  and  $t$ . We thus expect that the same result holds for the whole class of systems considered previously, satisfying the following two conditions:

- The kinetic energy  $T$  is quadratic in  $\dot{\mathbf{q}}$ ,

$$T = \frac{1}{2} \dot{\mathbf{q}} \cdot M(\mathbf{q}, t) \dot{\mathbf{q}} ;$$

here  $M(\mathbf{q}, t)$  is a matrix (the mass matrix) that must be real, symmetric and positive definite. Positive definiteness makes sure that in case of non-vanishing velocities  $T$  is positive.

- Following our definition of  $U$  we assume that the potential is independent of  $\dot{\mathbf{q}}$ , i.e.,

$$U = U(\mathbf{q}, t) .$$

The Lagrangian is thus of the form

$$L = \frac{1}{2} \dot{\mathbf{q}} \cdot M(\mathbf{q}, t) \dot{\mathbf{q}} - U(\mathbf{q}, t) .$$

The **momentum** can now be obtained as

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = M(\mathbf{q}, t) \dot{\mathbf{q}} \quad (5.10)$$

where we used that  $M(\mathbf{q}, t)$  is symmetric and applied the rule for gradients of quadratic functions from Section 3.5.2. Eq. (5.10) can be solved for  $\dot{\mathbf{q}}$ , to get

$$\dot{\mathbf{q}} = M(\mathbf{q}, t)^{-1} \mathbf{p} .$$

Here we have used that  $M$  is an invertible matrix. As shown in Linear Algebra, this follows from our conditions on  $M$ . (Idea of proof: Since  $M$  is real symmetric it can be diagonalised, i.e., it has eigenvalues and eigenvectors. To get the inverse we leave the eigenvectors as they are and invert the eigenvalues. This won't work if the eigenvalues are zero. However due to the positive definiteness of  $M$  the eigenvalues are positive and there are no difficulties.) If we use the definition of  $H$  and replace all  $\dot{\mathbf{q}}$  by  $M(\mathbf{q}, t)^{-1} \mathbf{p}$  the **Hamiltonian** is obtained as

$$\begin{aligned} H &= \mathbf{p} \cdot \dot{\mathbf{q}} - L \\ &= \mathbf{p} \cdot M(\mathbf{q}, t)^{-1} \mathbf{p} - \left( \frac{1}{2} (M(\mathbf{q}, t)^{-1} \mathbf{p}) \cdot \underbrace{M(\mathbf{q}, t) (M(\mathbf{q}, t)^{-1} \mathbf{p})}_{=\mathbf{p}} - U(\mathbf{q}, t) \right) \end{aligned}$$

Interchanging the two factors in the scalar product  $(M(\mathbf{q}, t)^{-1} \mathbf{p}) \cdot \mathbf{p}$ , we see that this product cancels half of the initial term  $\mathbf{p} \cdot M(\mathbf{q}, t)^{-1} \mathbf{p}$ . We thus obtain

$$H = \frac{1}{2} \mathbf{p} \cdot M(\mathbf{q}, t)^{-1} \mathbf{p} + U(\mathbf{q}, t)$$

where  $\frac{1}{2}\mathbf{p} \cdot M(\mathbf{q}, t)^{-1}\mathbf{p}$  is the kinetic energy. We have thus verified our expectation:

For systems of the type defined above the Hamiltonian is equal to  $E = T + U$ , written as a function of  $\mathbf{q}$ ,  $\mathbf{p}$  and  $t$ .

Let us now determine **Hamilton's equations**. For  $\dot{\mathbf{q}}$  we obtain

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}} = M(\mathbf{q}, t)^{-1}\mathbf{p},$$

again in line with our formula for  $\mathbf{p}$ . The formula for  $\dot{\mathbf{p}}$  is easier when written in components; it reads

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} = -\frac{1}{2}\mathbf{p} \cdot \frac{\partial M(\mathbf{q}, t)^{-1}}{\partial q_\alpha}\mathbf{p} - \frac{\partial U}{\partial q_\alpha}.$$

### Phase space

The essential difference between Lagrangian and Hamiltonian mechanics is that in Hamiltonian mechanics **coordinates and momenta are treated on equal footing**. Therefore, while Lagrange's equations were in configuration space, Hamilton's equation are equations in **phase space**. Phase space is the set  $\{(\mathbf{q}, \mathbf{p})\}$  of all allowed generalised coordinates and momenta. Each point in phase space is characterised by one  $\mathbf{q}$  and one  $\mathbf{p}$ , which are sometimes referred to as **canonical coordinates**. Since there is one generalised momentum for each coordinate, the dimension of phase space is  $2d$ , where  $d$  is the number of coordinates (or degrees of freedom).

Going from from configuration space to phase space has the following advantages:

- (Potentially) conserved quantities like momenta and the Hamiltonian play a much more central role than in Lagrangian mechanics. Hence **conservation laws** can be treated in a more systematic way, and can be used more efficiently to find solutions.
- An important advantage of Lagrangian mechanics was the freedom to choose any coordinates in phase space that we like. If one works in phase space, this **freedom in picking coordinates** is increased further. Instead of just going from Cartesian coordinates in configuration space to, say, polar coordinates we can make **variable transformations that mix coordinates and momenta**. This helps us to find variables in which a given problem becomes particularly simple.
- For many systems it is not possible to solve the equations of motion exactly. A famous example is three-body problem in which one considers the gravitational attraction between three bodies. If one wants to get trajectories of these systems one has to resort to **approximation methods** ("perturbation theory"). It turns out that these approximation methods can also be formulated more elegantly in phase space, using Hamiltonian mechanics.
- Phase space has interesting **geometrical** properties, with connections to pure mathematics (symplectic geometry, differential forms).

Outside classical mechanics, the Hamiltonian formulation also has close connections to **quantum theory**.

### Orders of equations

Lagrangian and Hamiltonian mechanics differ in the orders of the equations involved. **Lagrange's** equations contain the configuration-space coordinates  $\mathbf{q}$  and their first and second derivatives  $\dot{\mathbf{q}}$  and  $\ddot{\mathbf{q}}$ . The  $\mathbf{q}$ 's and  $\dot{\mathbf{q}}$ 's appear as arguments of the Lagrangian; the  $\ddot{\mathbf{q}}$ 's come into play because we take an additional time derivative in  $\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{q}}}$ . Hence Lagrange's equations are **second order differential equations in configuration space**.

In contrast Hamilton's equations are **first-order equations in phase space**: They specify  $\dot{\mathbf{q}}$  and  $\dot{\mathbf{p}}$  as functions of  $\mathbf{q}$  and  $\mathbf{p}$ , and there are no second derivatives. Going from Lagrange to Hamilton could thus be seen as a trade: reducing the order of equations to the first order while at the same time doubling the number of variables needed. This trick is often used in the theory of differential equations.

The first-order nature of Hamilton's equations has an important consequence: **If we know all phase-space variables, i.e., the coordinates  $\mathbf{q}(t)$  and the momenta  $\mathbf{p}(t)$  of a mechanical system at a time  $t$ , this determines the trajectories of the particles for all times.** Note that this is not true in configuration space: Given initial conditions only for the positions  $\mathbf{q}$  we would not be able to determine the motion of the system at later times.

To explain this statement, let us assume that we know  $\mathbf{q}(t)$  and  $\mathbf{p}(t)$ . Let us then try to find the coordinates and momenta after a short time interval  $\epsilon$ , i.e.  $\mathbf{q}(t + \epsilon)$  and  $\mathbf{p}(t + \epsilon)$ . Since  $\epsilon$  is small, we can approximate  $\mathbf{q}(t + \epsilon)$  by a Taylor expansion:

$$\mathbf{q}(t + \epsilon) \approx \mathbf{q}(t) + \dot{\mathbf{q}}(t)\epsilon .$$

Since Hamilton's equations are first order, we know  $\dot{\mathbf{q}}(t)$ . We can thus write

$$\mathbf{q}(t + \epsilon) \approx \mathbf{q}(t) + \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}(t), \mathbf{p}(t), t)\epsilon .$$

The same can be done for the momentum  $\mathbf{p}(t + \epsilon)$ :

$$\mathbf{p}(t + \epsilon) \approx \mathbf{p}(t) + \dot{\mathbf{p}}(t)\epsilon = \mathbf{p}(t) - \frac{\partial H}{\partial \mathbf{q}}(\mathbf{q}(t), \mathbf{p}(t), t)\epsilon .$$

Now this procedure can be carried further and further. Once we have  $\mathbf{q}(t + \epsilon), \mathbf{p}(t + \epsilon)$ , we can get coordinates and momenta  $\mathbf{q}(t + 2\epsilon), \mathbf{p}(t + 2\epsilon)$  after a further time step  $\epsilon$  by making another Taylor expansion. The changes of the coordinates and momenta are then given by  $\dot{\mathbf{q}}(t + \epsilon)\epsilon = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}(t + \epsilon), \mathbf{p}(t + \epsilon), t)\epsilon$  and  $\dot{\mathbf{p}}(t + \epsilon)\epsilon = -\frac{\partial H}{\partial \mathbf{q}}(\mathbf{q}(t + \epsilon), \mathbf{p}(t + \epsilon), t)\epsilon$ . We can continue like this, adding more and more time steps, until we have the trajectories for all times. A sketch of this procedure for systems with one-dimensional  $q$  and  $p$  is shown in Fig. 5.1.

For a fixed  $\epsilon \neq 0$ , our procedure gives a good approximation of  $\mathbf{q}$  and  $\mathbf{p}$  at later times. It can be used for example if we want to **numerically** determine the trajectory. However there is a small error since we dropped all higher-order terms in the Taylor expansion. If we want to prove that  $\mathbf{q}$  and  $\mathbf{p}$  at a given time determine the trajectories for all time, we therefore have to take the limit  $\epsilon \rightarrow 0$ .

Our observation has an important consequence: **Through each point in phase space there is only one unique trajectory.** Hence, if we take a picture of phase space like Fig. 5.1 and draw all possible trajectories, there will be no intersections.

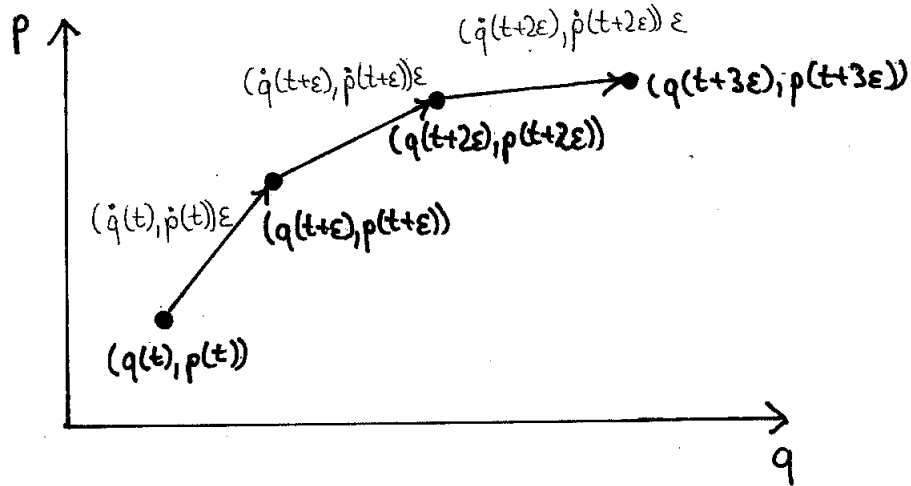


Figure 5.1: Stepwise solution of Hamilton's equations for one-dimensional  $q$  and  $p$ .

## 5.2 Conservation laws and Poisson brackets

In Lagrangian mechanics, we had seen that conservation laws could be read off from the Lagrangian: If  $L$  is independent of time, the generalised energy is conserved, if it is independent of one coordinate, the corresponding momentum is conserved. In Hamiltonian mechanics, these conservation laws are replaced by the following:

- If the Hamiltonian does not depend explicitly on time (i.e.  $\frac{\partial H}{\partial t} = 0$ ) it is conserved (i.e.  $\frac{dH}{dt} = 0$ ).

**Proof:** Using the chain rule and Hamilton's equations, we get

$$\begin{aligned} \frac{d}{dt}H(\mathbf{q}, \mathbf{p}, t) &= \underbrace{\frac{\partial H}{\partial \mathbf{q}}}_{=-\dot{\mathbf{p}}} \cdot \dot{\mathbf{q}} + \underbrace{\frac{\partial H}{\partial \mathbf{p}}}_{=\dot{\mathbf{q}}} \cdot \dot{\mathbf{p}} + \frac{\partial H}{\partial t} \\ &= -\dot{\mathbf{p}} \cdot \dot{\mathbf{q}} + \dot{\mathbf{q}} \cdot \dot{\mathbf{p}} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \end{aligned}$$

which is zero if  $H$  does not depend explicitly on  $t$ .

- If the Hamiltonian does not depend on a coordinate  $q_\alpha$  (i.e.  $\frac{\partial H}{\partial q_\alpha} = 0$ ) the corresponding momentum  $p_\alpha$  is conserved (i.e.  $\frac{dp_\alpha}{dt} = \dot{p}_\alpha = 0$ ).

**Proof:** This follows immediately from Hamilton's equation  $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$ .

- If the Hamiltonian does not depend on a momentum  $p_\alpha$  (i.e.  $\frac{\partial H}{\partial p_\alpha} = 0$ ) the corresponding coordinate  $q_\alpha$  is conserved (i.e.  $\frac{dq_\alpha}{dt} = \dot{q}_\alpha = 0$ ).

**Proof:** This follows immediately from Hamilton's equation  $\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$ .

Compared to Lagrangian mechanics we see that the first conservation law has a cleaner form (only  $H$  shows up, not  $L$  and  $h$ ) and that there is a new conservation law for coordinates; the latter appears because coordinates and momenta are treated on equal footing.

### How do arbitrary functions $F(\mathbf{q}, \mathbf{p}, t)$ change in time?

A conserved quantity need not be a coordinate or a momentum. Hence we should also find out when an arbitrary function  $F(\mathbf{q}, \mathbf{p}, t)$  is conserved and how it changes in time if it is not conserved. To do so we compute the total time derivative

$$\frac{dF}{dt} = \frac{\partial F}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial F}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} + \frac{\partial F}{\partial t}.$$

If we now use Hamilton's equations we obtain

$$\frac{dF}{dt} = \frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}} + \frac{\partial F}{\partial t}. \quad (5.11)$$

Here  $\frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}}$  is called the Poisson bracket of  $F$  and  $H$ . The general definition of the Poisson bracket is:

**Def.:** The **Poisson bracket** of two functions  $F(\mathbf{q}, \mathbf{p}, t)$  and  $G(\mathbf{q}, \mathbf{p}, t)$  is given by

$$\{F, G\} = \frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial G}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial G}{\partial \mathbf{q}} = \sum_{\gamma=1}^d \left( \frac{\partial F}{\partial q_{\gamma}} \frac{\partial G}{\partial p_{\gamma}} - \frac{\partial F}{\partial p_{\gamma}} \frac{\partial G}{\partial q_{\gamma}} \right).$$

With the Poisson bracket we can now restate Eq. (5.11) as follows:

The time derivative of a arbitrary function  $F(\mathbf{q}, \mathbf{p}, t)$  is given by

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}.$$

This leads to the following conservation law:

If  $F$  is independent of  $t$ , i.e.  $F = F(\mathbf{q}, \mathbf{p})$ , then it is a conserved quantity if its Poisson bracket with the Hamiltonian vanishes, i.e. if  $\{F, H\} = 0$ .

Poisson brackets allow to check easily whether a given quantity is conserved.

### Examples

#### a) Fundamental Poisson brackets

The simplest Poisson brackets are those between coordinates and momenta. They have the following form:

$$\begin{aligned} \{q_{\alpha}, q_{\beta}\} &= 0 \\ \{p_{\alpha}, p_{\beta}\} &= 0 \\ \{q_{\alpha}, p_{\beta}\} &= \delta_{\alpha\beta}. \end{aligned}$$

We see that all Poisson brackets between coordinates and momenta vanish except the one involving a coordinate and the corresponding momentum.

**Proof:** The first two formulas are trivial: When the functions  $F$  and  $G$  are both coordinates, the derivatives  $\frac{\partial F}{\partial \mathbf{p}}$  and  $\frac{\partial G}{\partial \mathbf{p}}$  in the definition of  $\{F, G\}$  vanish and the result is zero. Similarly when the functions  $F$  and  $G$  are both momenta, the derivatives  $\frac{\partial F}{\partial \mathbf{q}}$  and  $\frac{\partial G}{\partial \mathbf{q}}$  vanish and again the Poisson bracket is zero. The Poisson bracket involving coordinates and momenta

$$\{q_\alpha, p_\beta\} = \sum_{\gamma=1}^d \left( \frac{\partial q_\alpha}{\partial q_\gamma} \frac{\partial p_\beta}{\partial p_\gamma} - \frac{\partial q_\alpha}{\partial p_\gamma} \frac{\partial p_\beta}{\partial q_\gamma} \right).$$

is more complicated. To evaluate it we note that the derivative  $\frac{\partial q_\alpha}{\partial q_\gamma}$  is 1 if  $\alpha = \gamma$  and 0 otherwise. Hence it can be written as the Kronecker delta  $\delta_{\alpha\beta}$ . For the momenta we similarly have  $\frac{\partial p_\beta}{\partial p_\gamma} = \delta_{\beta\gamma}$ . In contrast the derivatives  $\frac{\partial q_\alpha}{\partial p_\gamma}$  and  $\frac{\partial p_\beta}{\partial q_\gamma}$  are always zero. We thus obtain

$$\{q_\alpha, p_\beta\} = \sum_{\gamma=1}^d \delta_{\alpha\gamma} \delta_{\beta\gamma}.$$

Here the summand is 1 if all coefficients coincide, i.e., if  $\alpha = \beta = \gamma$ . Otherwise it is zero. Hence for  $\alpha \neq \beta$  all summands are zero, while for  $\alpha = \beta$  one summand is 1. The sum thus yields

$$\{q_\alpha, p_\beta\} = \delta_{\alpha\beta}$$

as required.

## b) Components of the angular momentum

We now evaluate the Poisson brackets involving the components of the angular momentum

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix},$$

i.e.,

$$\begin{aligned} l_x &= yp_z - zp_y \\ l_y &= zp_x - xp_z \\ l_z &= xp_y - yp_x. \end{aligned}$$

The Poisson bracket of the first two components is

$$\{l_x, l_y\} = \frac{\partial l_x}{\partial x} \frac{\partial l_y}{\partial p_x} + \frac{\partial l_x}{\partial y} \frac{\partial l_y}{\partial p_y} + \frac{\partial l_x}{\partial z} \frac{\partial l_y}{\partial p_z} - (l_x \leftrightarrow l_y)$$

where  $(l_x \leftrightarrow l_y)$  indicates the last three summands of the Poisson bracket for which the roles of  $l_x$  and  $l_y$  are interchanged. Now we can use that  $\frac{\partial l_x}{\partial x} = 0$ ,  $\frac{\partial l_y}{\partial y} = 0$ ,  $\frac{\partial l_x}{\partial p_x} = 0$  and  $\frac{\partial l_y}{\partial p_y} = 0$ . This leaves only two summands, and we get

$$\{l_x, l_y\} = \frac{\partial l_x}{\partial z} \frac{\partial l_y}{\partial p_z} - \frac{\partial l_y}{\partial z} \frac{\partial l_x}{\partial p_z} = (-p_y)(-x) - p_x y = l_z.$$

We thus see that the Poisson bracket of the first two components of the angular momentum yields the third component.



### Properties of Poisson brackets

We will now derive a few rules that simplify the computation of more complicated Poisson brackets. Here  $F, F_1, F_2, G, H$  are functions of  $\mathbf{q}, \mathbf{p}$  (and possibly time), and  $a_1$  and  $a_2$  are numbers.

1. Linearity:

$$\{a_1 F_1 + a_2 F_2, G\} = a_1 \{F_1, G\} + a_2 \{F_2, G\}$$

2. Antisymmetry:

$$\{F, G\} = -\{G, F\}$$

3. Product rule:

$$\{FG, H\} = F\{G, H\} + G\{F, H\}$$

4. Jacobi identity:

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$$

These rules imply that we should think of  $\{F, G\}$  as a kind of cross product of phase-space functions. It should be a product since the definition involves products of derivatives of  $F$  and  $G$ ; also the linearity property is expected to hold for products. However like the cross product, this product is an antisymmetric one, since it changes sign if the two factors are interchanged.

Let us now **prove** these rules:

1. Linearity can be checked trivially if we insert  $a_1 F_1 + a_2 F_2$  into the definition of the Poisson bracket.
2. To prove antisymmetry we rearrange the terms in  $\{F, G\}$  and see that the result coincides with  $-\{G, F\}$ :

$$\begin{aligned} \{F, G\} &= \frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial G}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial G}{\partial \mathbf{q}} \\ &= - \left( \frac{\partial G}{\partial \mathbf{q}} \cdot \frac{\partial F}{\partial \mathbf{p}} - \frac{\partial G}{\partial \mathbf{p}} \cdot \frac{\partial F}{\partial \mathbf{q}} \right) = -\{G, F\}. \end{aligned}$$

3. To prove the product rule for the Poisson bracket, we use the product rule for derivatives:

$$\begin{aligned} \{FG, H\} &= \frac{\partial(FG)}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial(FG)}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}} \\ &= F \frac{\partial G}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} + G \frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - F \frac{\partial G}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}} - G \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}} \\ &= F\{G, H\} + G\{F, H\} \end{aligned}$$

4. The proof of Jacobi's identity is a bit messy. It is listed below for completeness:

For simplicity we'll consider the case of one degree of freedom. The calculation in higher dimensions is essentially the same; the only complication is keeping track of indices.

Let  $F = F(q, p)$ ,  $G = G(q, p)$  and  $H = H(q, p)$  be functions on phase space. We wish to show that

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0. \quad (5.12)$$

Let's evaluate the first term explicitly. We have that

$$\{F, G\} = F_q G_p - F_p G_q,$$

where  $F_q = \partial F / \partial q$ , and similarly  $F_p$ ,  $G_q$  and  $G_p$ . Then

$$\begin{aligned} \{\{F, G\}, H\} &= \frac{\partial \{F, G\}}{\partial q} H_p - \frac{\partial \{F, G\}}{\partial p} H_q \\ &= \frac{\partial (F_q G_p - F_p G_q)}{\partial q} H_p - \frac{\partial (F_q G_p - F_p G_q)}{\partial p} H_q, \end{aligned}$$

or

$$\{\{F, G\}, H\} = (F_{qq} G_p + F_q G_{pq} - F_{pq} G_q - F_p G_{qq}) H_p - (F_{qp} G_p + F_q G_{pp} - F_{pp} G_q - F_p G_{qp}) H_q, \quad (5.13)$$

where  $F_{qq} = \partial^2 F / \partial q \partial q$ , and similarly for  $F_{qp}$ ,  $F_{pp}$ ,  $G_{qq}$ , etc. Note that  $F_{qp} = F_{pq}$ , i.e., the ordering of mixed partials doesn't matter; similarly for  $G_{qp}$  and  $H_{qp}$ . Similarly, for the other two terms in (5.12),

$$\{\{G, H\}, F\} = (G_{qq} H_p + G_q H_{pq} - G_{pq} H_q - G_p H_{qq}) F_p - (G_{qp} H_p + G_q H_{pp} - G_{pp} H_q - G_p H_{qp}) F_q, \quad (5.14)$$

and

$$\{\{H, F\}, G\} = (H_{qq} F_p + H_q F_{pq} - H_{pq} F_q - H_p F_{qq}) G_p - (H_{qp} F_p + H_q F_{pp} - H_{pp} F_q - H_p F_{qp}) G_q. \quad (5.15)$$

Now it is simply a matter of adding together terms from (5.13), (5.14) and (5.15) to see that everything cancels. For example, consider the term  $F_{qq} G_p H_p$ . This appears in both (5.13) and (5.15), but with opposite signs. Likewise, you can check that every other term appears in just two of the expressions (5.13), (5.14) and (5.15) but with opposite signs.

In the rules 1, 3 and 4 above always the first argument of the Poisson bracket was something interesting (a sum, a product or another Poisson bracket). Completely **analogous rules hold if the second argument is a sum, product or Poisson bracket**. They can all be proven from the rules already derived, if we use the antisymmetry rule to interchange the two arguments.

### Example

To illustrate the above rules let us evaluate the Poisson bracket  $\{p_y, l_z\}$ :

$$\begin{aligned} \{p_y, l_z\} &= \{p_y, xp_y - yp_x\} \\ &= \{p_y, xp_y\} - \{p_y, yp_x\} \\ &= \{p_y, x\} p_y + \{p_y, p_y\} x - \{p_y, y\} p_x - \{p_y, p_y\} y \\ &= -\{p_y, y\} p_x \\ &= \{y, p_y\} p_x \\ &= p_x \end{aligned}$$

Here we used linearity to get from the first to the second line, and the product rule to get from the second to the third line. The third line then contains only Poisson brackets of coordinates and momenta, which vanish apart from the term  $-\{p_y, y\}$ . Finally we used antisymmetry and the fundamental Poisson brackets to get  $-\{p_y, y\} = \{y, p_y\} = 1$ .

**Example (optional)**

Consider a phase space with one coordinate  $q$  and one momentum  $p$ . Let us evaluate the Poisson bracket of  $q^\alpha \cos(\beta p)$  and  $q^\alpha \sin(\beta p)$ :

$$\begin{aligned}
 \{q^\alpha \cos(\beta p), q^\alpha \sin(\beta p)\} &= q^\alpha \{\cos(\beta p), q^\alpha \sin(\beta p)\} + \cos(\beta p) \{q^\alpha, q^\alpha \sin(\beta p)\} \\
 &= q^\alpha q^\alpha \{\cos(\beta p), \sin(\beta p)\} + q^\alpha \sin(\beta p) \{\cos(\beta p), q^\alpha\} + \cos(\beta p) q^\alpha \{q^\alpha, \sin(\beta p)\} + \cos(\beta p) \sin(\beta p) \{q^\alpha, q^\alpha\} \\
 &= 0 + q^\alpha \sin(\beta p) \left( -\frac{\partial \cos(\beta p)}{\partial p} \frac{\partial q^\alpha}{\partial q} \right) + \cos(\beta p) q^\alpha \left( \frac{\partial q^\alpha}{\partial q} \frac{\partial \sin(\beta p)}{\partial p} \right) + 0 \\
 &= \alpha \beta \sin^2(\beta p) q^{2\alpha-1} + \alpha \beta \cos^2(\beta p) q^{2\alpha-1} \\
 &= \alpha \beta q^{2\alpha-1}
 \end{aligned}$$

In the first two lines we used the product rule. Then the derivatives were evaluated.

**Applications**

As mentioned above, Poisson brackets can be used to **check whether a quantity is conserved**. Moreover Poisson brackets can be used to **generate new conservation laws**. This is due to Poisson's theorem:

**Poisson's theorem:** If two phase-space functions  $A(\mathbf{q}, \mathbf{p})$  and  $B(\mathbf{q}, \mathbf{p})$  are conserved then their Poisson bracket  $\{A, B\}$  is conserved as well.

**Proof:** We assume that  $A$  and  $B$  are conserved, i.e., that their Poisson bracket with the Hamiltonian  $H$  vanishes. We then use the Jacobi identity to evaluate the Poisson bracket of  $\{A, B\}$  with  $H$ . The result is zero:

$$\{\{A, B\}, H\} = -\underbrace{\{\{B, H\}, A\}}_{=0} - \underbrace{\{\{H, A\}, B\}}_{=0} = 0.$$

Hence the Poisson bracket of  $\{A, B\}$  with  $H$  vanishes and  $\{A, B\}$  is a conserved quantity.

We can thus get additional conserved quantities of a system by computing the Poisson brackets of the conserved quantities we already know. For example, if we know that  $l_x$  and  $l_y$  are conserved then their Poisson bracket  $\{l_x, l_y\} = l_z$  must be conserved as well, and if  $p_y$  and  $l_z$  are conserved the same must hold for  $\{p_y, l_z\} = p_x$ . However computing Poisson brackets need not always give something new; the Poisson bracket  $\{A, B\}$  may also be linearly dependent of  $A$  and  $B$  or trivial (say, zero).

**5.3 Canonical transformations**

A strong advantage of Lagrangian mechanics was that the equations of motion have the same form in all systems of coordinate. This freedom of picking coordinates becomes even larger in Hamiltonian mechanics: Now we can perform coordinate transformations in phase space, including **transformations that mix generalised coordinates and generalised momenta**. Not all of these transformations will be useful, but only those in which the basic ingredients of Hamiltonian mechanics (such as the Poisson brackets) remain the same. These transformations are called **canonical transformations** and defined as follows:

**Def.:** Consider a transformation of coordinates in phase space

$$\begin{aligned} Q_\alpha &= Q_\alpha(\mathbf{q}, \mathbf{p}) \\ P_\alpha &= P_\alpha(\mathbf{q}, \mathbf{p}) \end{aligned}$$

and the Poisson bracket defined by derivatives w.r.t.  $\mathbf{Q}$  and  $\mathbf{P}$

$$\{F, G\}_{\mathbf{Q}, \mathbf{P}} \equiv \frac{\partial F}{\partial \mathbf{Q}} \cdot \frac{\partial G}{\partial \mathbf{P}} - \frac{\partial F}{\partial \mathbf{P}} \cdot \frac{\partial G}{\partial \mathbf{Q}}.$$

The transformation is called **canonical** if the Poisson bracket  $\{F, G\}_{\mathbf{Q}, \mathbf{P}}$  coincides with  $\{F, G\} = \frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial G}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial G}{\partial \mathbf{q}}$  if we insert for  $F$  and  $G$  the coordinates and momenta,

$$\begin{aligned} \{q_\alpha, q_\beta\}_{\mathbf{Q}, \mathbf{P}} &= \{q_\alpha, q_\beta\} = 0 \\ \{p_\alpha, p_\beta\}_{\mathbf{Q}, \mathbf{P}} &= \{p_\alpha, p_\beta\} = 0 \\ \{q_\alpha, p_\beta\}_{\mathbf{Q}, \mathbf{P}} &= \{q_\alpha, p_\beta\} = \delta_{\alpha\beta}. \end{aligned} \tag{5.16}$$

In other words, canonical transformations preserve the fundamental Poisson brackets.

**Example:** Consider a system with just one coordinate  $q$  and one momentum  $p$ . We can now define a transformation that just interchanges the coordinate and the momentum and flips a sign:

$$\begin{aligned} Q &= -p \\ P &= q \end{aligned}$$

This transformation is canonical because

$$\begin{aligned} \{q, q\}_{Q, P} &= 0 \\ \{p, p\}_{Q, P} &= 0 \\ \{q, p\}_{Q, P} &= \frac{\partial q}{\partial Q} \frac{\partial p}{\partial P} - \frac{\partial q}{\partial P} \frac{\partial p}{\partial Q} = \frac{\partial P}{\partial Q} \frac{\partial(-Q)}{\partial P} - \frac{\partial P}{\partial P} \frac{\partial(-Q)}{\partial Q} = 1. \end{aligned}$$

Here the first two lines are trivial as the Poisson bracket of any quantity with itself is zero.

**Thm.:** If the transformation leading from  $\mathbf{q}, \mathbf{p}$  to  $\mathbf{Q}, \mathbf{P}$  is canonical we have  $\{F, G\}_{\mathbf{Q}, \mathbf{P}} = \{F, G\}$  for all phase-space functions  $F$  and  $G$ . In other words, canonical transformations preserve all Poisson brackets.

**Proof:** We assume that (5.16) holds and try to show that  $\{F, G\}_{\mathbf{Q}, \mathbf{P}} = \{F, G\}$ .

We get

$$\begin{aligned}
\{F, G\}_{Q, P} &= \frac{\partial F}{\partial \mathbf{Q}} \cdot \frac{\partial G}{\partial \mathbf{P}} - \frac{\partial F}{\partial \mathbf{P}} \cdot \frac{\partial G}{\partial \mathbf{Q}} \\
&= \sum_{\alpha=1}^d \left( \frac{\partial F}{\partial q_{\alpha}} \frac{\partial q_{\alpha}}{\partial \mathbf{Q}} + \frac{\partial F}{\partial p_{\alpha}} \frac{\partial p_{\alpha}}{\partial \mathbf{Q}} \right) \cdot \sum_{\beta=1}^d \left( \frac{\partial G}{\partial q_{\beta}} \frac{\partial q_{\beta}}{\partial \mathbf{P}} + \frac{\partial G}{\partial p_{\beta}} \frac{\partial p_{\beta}}{\partial \mathbf{P}} \right) - \left[ \frac{\partial}{\partial \mathbf{Q}} \leftrightarrow \frac{\partial}{\partial \mathbf{P}} \right] \\
&= \sum_{\alpha=1}^d \sum_{\beta=1}^d \left[ \frac{\partial F}{\partial q_{\alpha}} \frac{\partial G}{\partial q_{\beta}} \underbrace{\left( \frac{\partial q_{\alpha}}{\partial \mathbf{Q}} \cdot \frac{\partial q_{\beta}}{\partial \mathbf{P}} - \frac{\partial q_{\alpha}}{\partial \mathbf{P}} \cdot \frac{\partial q_{\beta}}{\partial \mathbf{Q}} \right)}_{=\{q_{\alpha}, q_{\beta}\}_{Q, P}=0} \right. \\
&\quad + \frac{\partial F}{\partial q_{\alpha}} \frac{\partial G}{\partial p_{\beta}} \underbrace{\left( \frac{\partial q_{\alpha}}{\partial \mathbf{Q}} \cdot \frac{\partial p_{\beta}}{\partial \mathbf{P}} - \frac{\partial q_{\alpha}}{\partial \mathbf{P}} \cdot \frac{\partial p_{\beta}}{\partial \mathbf{Q}} \right)}_{=\{q_{\alpha}, p_{\beta}\}_{Q, P}=\delta_{\alpha\beta}} \\
&\quad + \frac{\partial F}{\partial p_{\alpha}} \frac{\partial G}{\partial q_{\beta}} \underbrace{\left( \frac{\partial p_{\alpha}}{\partial \mathbf{Q}} \cdot \frac{\partial q_{\beta}}{\partial \mathbf{P}} - \frac{\partial p_{\alpha}}{\partial \mathbf{P}} \cdot \frac{\partial q_{\beta}}{\partial \mathbf{Q}} \right)}_{=\{p_{\alpha}, q_{\beta}\}_{Q, P}=-\{q_{\beta}, p_{\alpha}\}_{Q, P}=-\delta_{\alpha\beta}} \\
&\quad \left. + \frac{\partial F}{\partial p_{\alpha}} \frac{\partial G}{\partial p_{\beta}} \underbrace{\left( \frac{\partial p_{\alpha}}{\partial \mathbf{Q}} \cdot \frac{\partial p_{\beta}}{\partial \mathbf{P}} - \frac{\partial p_{\alpha}}{\partial \mathbf{P}} \cdot \frac{\partial p_{\beta}}{\partial \mathbf{Q}} \right)}_{=\{p_{\alpha}, p_{\beta}\}_{Q, P}=0} \right] \\
&= \sum_{\alpha=1}^d \left( \frac{\partial F}{\partial q_{\alpha}} \frac{\partial G}{\partial p_{\alpha}} - \frac{\partial F}{\partial p_{\alpha}} \frac{\partial G}{\partial q_{\alpha}} \right) \\
&= \{F, G\} .
\end{aligned}$$

In the first line, I simply wrote down the definition of the Poisson bracket. In the second line, the first two derivatives were evaluated using the chain rule (i.e.,  $\frac{\partial F}{\partial \mathbf{Q}} = \sum_{\alpha=1}^d \left( \frac{\partial F}{\partial q_{\alpha}} \frac{\partial q_{\alpha}}{\partial \mathbf{Q}} + \frac{\partial F}{\partial p_{\alpha}} \frac{\partial p_{\alpha}}{\partial \mathbf{Q}} \right)$ ), and it was noted that  $\frac{\partial F}{\partial \mathbf{P}} \cdot \frac{\partial G}{\partial \mathbf{Q}}$  can be obtained from  $\frac{\partial F}{\partial \mathbf{Q}} \cdot \frac{\partial G}{\partial \mathbf{P}}$  by interchanging the derivatives w.r.t.  $\mathbf{Q}$  and  $\mathbf{P}$ . Then the derivatives were grouped together to obtain the fundamental Poisson brackets.

### Hamilton's equations

Canonical transformation also preserve Hamilton's equations.

**Thm.:** If the transformation  $\mathbf{q}, \mathbf{p} \rightarrow \mathbf{Q}, \mathbf{P}$  is time independent and canonical, we have

$$\begin{aligned}
\dot{Q}_{\alpha} &= \frac{\partial H}{\partial P_{\alpha}} \\
\dot{P}_{\alpha} &= -\frac{\partial H}{\partial Q_{\alpha}},
\end{aligned}$$

i.e., if we express the Hamiltonian in terms of  $\mathbf{Q}, \mathbf{P}$  and take derivatives w.r.t. these quantities we get Hamilton's equations just as for  $\mathbf{q}, \mathbf{p}$ .

**Proof:** We have

$$\begin{aligned}
 \dot{Q}_\alpha &= \{Q_\alpha, H\} + \underbrace{\frac{\partial Q_\alpha}{\partial t}}_{=0} \\
 &= \{Q_\alpha, H\}_{\mathbf{Q}, \mathbf{P}} \\
 &= \sum_{\beta=1}^d \left( \underbrace{\frac{\partial Q_\alpha}{\partial Q_\beta}}_{=\delta_{\alpha\beta}} \frac{\partial H}{\partial P_\beta} - \underbrace{\frac{\partial Q_\alpha}{\partial P_\beta}}_{=0} \frac{\partial H}{\partial Q_\beta} \right) \\
 &= \frac{\partial H}{\partial P_\alpha}
 \end{aligned}$$

Here I used that the total time derivative of every phase space function can be written as a Poisson bracket with the Hamiltonian plus a partial time derivative. In our case the partial time derivative vanishes because the  $Q_\alpha$ 's do not depend explicitly on time. Then the Poisson bracket was written in terms of  $\mathbf{Q}, \mathbf{P}$ , and in this form it could be evaluated easily. Similarly we obtain

$$\begin{aligned}
 \dot{P}_\alpha &= \{P_\alpha, H\} + \underbrace{\frac{\partial P_\alpha}{\partial t}}_{=0} \\
 &= \{P_\alpha, H\}_{\mathbf{Q}, \mathbf{P}} \\
 &= \sum_{\beta=1}^d \left( \frac{\partial P_\alpha}{\partial Q_\beta} \frac{\partial H}{\partial P_\beta} - \underbrace{\frac{\partial P_\alpha}{\partial P_\beta}}_{=\delta_{\alpha\beta}} \frac{\partial H}{\partial Q_\beta} \right) \\
 &= -\frac{\partial H}{\partial Q_\alpha} .
 \end{aligned}$$

### Example

To illustrate canonical transformations, let us consider the one-dimensional harmonic oscillator. If we set the mass equal to 1, the harmonic oscillator has the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2 .$$

Since  $H$  does not depend explicitly on time, it is a conserved quantity. This means that in phase space the particles can only move on trajectories for which  $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2 = \text{const}$ , i.e., on ellipses (or circles if  $\omega = 1$ ), see Fig. 5.2. But we still have to determine how fast the particles go along these ellipses.

The **usual** way to solve this problem would be to use Hamilton's equations with variables  $q$  and  $p$ . One thus gets

$$\begin{aligned}
 \dot{q} &= \frac{\partial H}{\partial p} = p \\
 \dot{p} &= -\frac{\partial H}{\partial q} = -\omega^2q .
 \end{aligned}$$

This yields  $\ddot{q} = \dot{p} = -\omega^2q$  and thus  $q = A \sin(\omega t + \phi)$  where  $A$  and  $\phi$  are constants. For  $p$  we obtain  $p = \dot{q} = A\omega \cos(\omega t + \phi)$ .

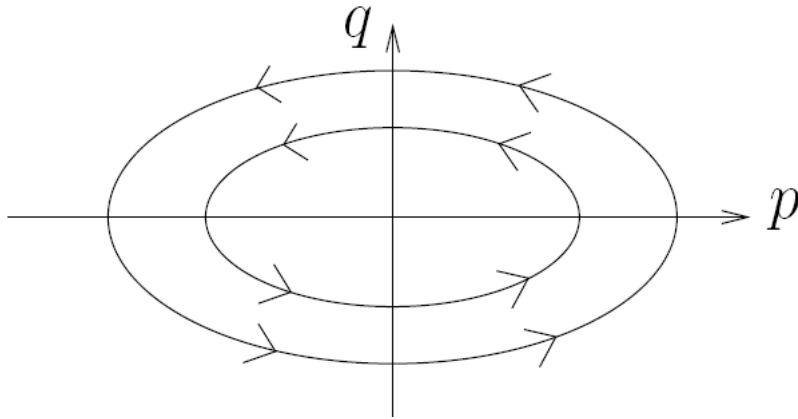


Figure 5.2: The harmonic oscillator in phase space.

**Alternatively** we can make a coordinate transformation inspired by the form of the curves. If the phase-space curves were circles, it would be convenient to use **polar coordinates in phase space**. For ellipses one takes

$$\begin{aligned} p &= \sqrt{2I\omega} \cos \theta \\ q &= \sqrt{\frac{2I}{\omega}} \sin \theta . \end{aligned} \quad (5.17)$$

(One easily sees that for  $\omega = 1$ , this boils down to polar coordinates with a radius  $r = \sqrt{2I}$ , and we would have  $H = \frac{1}{2}p^2 + \frac{1}{2}q^2 = \frac{1}{2}r^2 = I$ .) The coordinate  $I$  can assume arbitrary positive values. Since  $q, p$  remain the same if the  $\theta$  is increased by  $2\pi$  the definition range of  $\theta$  should be taken as  $[0, 2\pi)$ . Now one of these variables should be interpreted as a generalised coordinate and the other one as a generalised momentum. It turns out that the appropriate choice is to take  $\theta$  (the “**angle**”) as the coordinate and  $I$  (the “**action**”) as the momentum.

We have to show that this transformation is **canonical**. This can be done by checking that the fundamental Poisson bracket  $\{q, p\}_{\theta, I}$  is 1 (note that  $\{q, q\}_{\theta, I} = 0$  and  $\{p, p\}_{\theta, I} = 0$  are trivial),

$$\begin{aligned} \{q, p\}_{\theta, I} &= \frac{\partial q}{\partial \theta} \frac{\partial p}{\partial I} - \frac{\partial q}{\partial I} \frac{\partial p}{\partial \theta} \\ &= \sqrt{\frac{2I}{\omega}} \cos \theta \sqrt{\frac{\omega}{2I}} \cos \theta - \sqrt{\frac{1}{2I\omega}} \sin \theta \left( -\sqrt{2I\omega} \right) \sin \theta \\ &= 1 . \end{aligned}$$

Expressed in terms of  $\theta$  and  $I$  the **Hamiltonian** now becomes

$$\begin{aligned} H &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \\ &= I\omega \cos^2 \theta + I\omega \sin^2 \theta \\ &= I\omega , \end{aligned}$$

and **Hamilton's equations** read

$$\begin{aligned}\dot{\theta} &= \frac{\partial H}{\partial I} = \omega \\ \dot{I} &= -\frac{\partial H}{\partial \theta} = 0\end{aligned}$$

We have thus managed to express the Hamiltonian in terms of only one variable,  $I$ . Since the coordinate  $\theta$  does not show up, the corresponding momentum  $I$  has become a conserved quantity – indicating that the particle stays on an ellipse in phase space. Also we have seen that  $\theta$  increases with constant speed  $\omega$ . However once  $\theta$  reaches  $2\pi$ ,  $p$  and  $q$  become the same as for  $\theta = 0$ , i.e.  $\theta$  is reset to zero. If we draw the phase-space trajectories in a coordinate system parametrised by  $\theta$  and  $I$ , the trajectories thus look like in Fig. 5.3: The ellipses from Fig. 5.2 have been turned into straight lines.

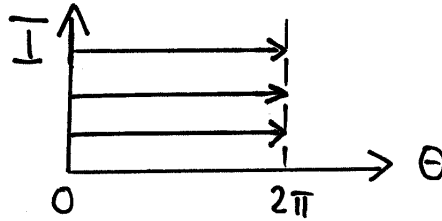


Figure 5.3: The harmonic oscillator in action-angle variables.

### Integrable systems

There are many more systems for which one can find a canonical transformation such that the motion in phase space becomes trivial (going along straight lines):

**Def.:** A system is called **integrable** if there is a canonical transformation

$$q, p \rightarrow \theta, I$$

(with  $q, p, \theta, I$  denoting  $d$ -dimensional vectors) such that the Hamiltonian can be written as a function of  $I$  only:

$$H = H(I).$$

As for the harmonic oscillator the components of  $\theta$  are called **angle variables** and the components of  $I$  are called **action variables**.

For integrable systems Hamilton's equations read

$$\begin{aligned}\dot{\theta}_\alpha &= \frac{\partial H}{\partial I_\alpha} \\ \dot{I}_\alpha &= -\frac{\partial H}{\partial \theta_\alpha} = 0\end{aligned}$$

The second equation means that all  $I_\alpha$ 's are conserved quantities. Since in the first equation  $\dot{\theta} = \frac{\partial H}{\partial I_\alpha}$  depends only on these  $I_\alpha$ 's it must be constant as well and  $\theta_\alpha$



increases linearly. Thus, expressed in terms of  $\boldsymbol{\theta}, \mathbf{I}$  the phase-space motion indeed goes along straight lines.

Now which systems are integrable? It is important that there must be  $d$  different conserved quantities  $I_\alpha$ . For a one-dimensional system one conservation law (for the energy) is enough, in higher-dimensional systems we need more. Also it must be possible to pick these quantities as momenta after a suitable canonical transformation. Hence they must be mutually independent and their Poisson brackets  $I_\alpha, I_\beta$  must vanish.

Roughly speaking, integrable systems are “nice” systems which satisfy enough conservation laws to allow for an analytical solution. Most systems considered in this lecture are of this type. For example the spherical pendulum is described by two generalised coordinates and has two conserved quantities (the energy and the angular momentum), which we had already used to obtain an analytical solution.