$\underline{\text{Im}T}$: $\text{Im}T = \{T(x_1,x_2,x_3,x_4) = (x_1+x_2,\ x_2-x_3,\ x_1+x_4) : x_1,x_2,x_3,x_4 \in \mathbb{R}\} =$

$= \{x_1\cdot(1,0,1) + x_2\cdot(1,1,0) + x_3\cdot(0,-1,0) + x_4\cdot(0,0,1) : x_1,x_2,x_3,x_4 \in \mathbb{R}\}.$

Уочимо, вектори $(1,0,1)$, $(0,-1,0)$ и $(0,0,1)$ су линеарно независни, па чине базу $\mathbb{R}^3$:

$$\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{-1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \Rightarrow \boxed{\text{Im}T = \mathbb{R}^3}.$$

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7. Да ли су вектори $\underset{g_1}{(2,4,-1,-2,0)}$, $\underset{g_2}{(5,-1,0,2,0)}$ и $\underset{g_3}{(0,1,-2,1,0)}$ лин. независни?

$$\begin{bmatrix} 2 & 4 & -1 & -2 & 0 \\ 5 & -1 & 0 & 2 & 0 \\ 0 & \boxed{1} & -2 & 1 & 0 \end{bmatrix}_{(-4)} \sim \begin{bmatrix} 2 & 0 & 7 & -6 & 0 \\ 5 & 0 & -2 & \boxed{3} & 0 \\ 0 & \boxed{1} & -2 & 1 & 0 \end{bmatrix}_{\cdot 2} \sim \begin{bmatrix} 12 & 0 & \boxed{3} & 0 & 0 \\ 5 & 0 & -2 & \boxed{3} & 0 \\ 0 & \boxed{1} & -2 & 1 & 0 \end{bmatrix}$$

Вектори јесу линеарно независни.

Дефинишемо линеарно пресликавање на канонској бази $e=[e_1,e_2,e_3,e_4]$:

$L(e_1)=g_1$, $L(e_2)=g_2$, $L(e_3)=g_3$, $L(e_4)=0$. Ово пресликавање задовољава $\text{Im}L = \text{Lin}(g_1,g_2,g_3)$

8. $L: M_2(\mathbb{R}) \to M_2(\mathbb{R})$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $L(X) = \begin{bmatrix} 9a-2b+5c & 8a-b+5d \\ 4c-2d & 8c-6d \end{bmatrix}$

$L(X) = \lambda\cdot X$: $\begin{bmatrix} 9a-2b+5c & 8a-b+5d \\ 4c-2d & 8c-6d \end{bmatrix} = \lambda\cdot\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{aligned} (9-\lambda)a - 2b + 5c &= 0 \\ 8\boxed{a} - (1+\lambda)b + 5d &= 0 \quad \Big|\ \tfrac{\lambda-9}{8} \\ (4-\lambda)c - 2d &= 0 \\ 8c - (6+\lambda)d &= 0 \end{aligned}$$

$$\begin{aligned} 8\boxed{a} - (1+\lambda)b + 5d &= 0 \\ -\left[(1+\lambda)\tfrac{\lambda-9}{8} + 2\right]b + 5c + \tfrac{5}{8}(\lambda-9)d &= 0 \\ (4-\lambda)c - 2d &= 0 \\ 8c - (6+\lambda)d &= 0 \end{aligned}$$

1° $\frac{1}{8}(1+\lambda)(\lambda-9) + 2 = \frac{1}{8}\lambda^2 - \lambda + \frac{7}{8} = 0$

$\lambda_{1,2} = \dfrac{1 \pm \sqrt{1-\frac{7}{16}}}{\frac{1}{4}} = \dfrac{1 \pm \sqrt{\frac{9}{16}}}{\frac{1}{4}} = 4 \pm 3 = \begin{cases} 1 \\ 7 \end{cases}$

$\underline{\lambda_1 = 1}$:

$$\begin{aligned} 8\boxed{a} - 2b + 5d &= 0 \\ 5\boxed{c} - 5d &= 0 \quad \Big|\ \left(-\tfrac{3}{5}\right)\left(-\tfrac{5}{8}\right) \\ 3c - 2d &= 0 \\ 8c - \cancel{7}d &= 0 \end{aligned}$$

$\vdots$

$1\cdot d = 0 \Rightarrow \boxed{c=d=0,\ b\in\mathbb{R},\ a=\frac{b}{4}}$

$b \neq 0$

$\lambda_1 = 1$ сопств. вред.; $b\cdot(1,4,0,0)$ сопствени вектор

$\lambda_2 = 7$: $8\boxed{a} - 8b \quad + 5d = 0$

$5c - \frac{5}{4}d = 0$

$-3\boxed{c} - 2d = 0 \quad \frac{5}{3} \quad \frac{8}{3}$

$8c - 13d = 0$

$\vdots$

$k \cdot d = 0 \Rightarrow \boxed{c = d = 0}, \boxed{b \in \mathbb{R}}, \boxed{a = b}$

$k \neq 0$

$\lambda_2 = 7$ сопствена вредност, $b \cdot (1,1,0,0)$ сопствени вектор

$\lambda \notin \{1,7\}$: $8\boxed{a} - (1+\lambda)b \quad + 5d = 0$

$-[(1+\lambda)\frac{\lambda-9}{8} + 2]\boxed{b} + 5c + \frac{5}{8}(\lambda-9)d = 0$

$(4-\lambda)c - 2d = 0$

$8\boxed{c} - (6+\lambda)d = 0 \quad \frac{(\lambda-4)}{8}$

$\vdots$

$8\boxed{c} - (6+\lambda)d = 0$

$[\frac{1}{8}(-\lambda^2 - 2\lambda + 24) - 2]d = 0 \; / \cdot (-8)$

$(\lambda^2 + 2\lambda - 8) \cdot d = 0$

$\lambda_{3,4} = \frac{-2 \pm \sqrt{4+32}}{2} = \begin{cases} -4 \\ 2 \end{cases}$

$\lambda_3 = -4$: $\boxed{d \in \mathbb{R}}$, $\boxed{c = \frac{1}{4}d}$, $-[(-3) \cdot \frac{-13}{8} + 2]b + \frac{5}{4}d + \frac{5 \cdot (-13)}{8}d = 0$

$-\frac{55}{8}b + \frac{5}{4}d - \frac{55}{8}d = 0 \Rightarrow \boxed{b = (-\frac{8}{55}) \cdot \frac{45}{8}d = -\frac{9}{11}d}$

$8a + 3b \quad + 5d = 0$

$\cancel{(a = \frac{27}{11}d)}$ $\boxed{a = (\frac{27}{11}d - 5d) \cdot \frac{1}{8} = -\frac{28}{11 \cdot 8}d = -\frac{7}{22}d}$

Сопствени вектор: $d \cdot (-\frac{7}{22}, -\frac{9}{11}, \frac{1}{4}, 1)$

$\lambda_4 = 2$: $\boxed{d \in \mathbb{R}}$, $\boxed{c = d}$, $-[3 \cdot (-\frac{7}{8}) + 2]b + 5d - \frac{35}{8}d = 0$

$\frac{5}{8}b + \frac{5}{8}d = 0 \Rightarrow \boxed{b = -d}$

$8a + 3b \quad + 5d = 0$

$8a + 2d = 0$

$\boxed{a = \frac{1}{4}d}$

Сопствени вектор: $d \cdot (\frac{1}{4}, -1, 1, 1)$

$\lambda \notin \{-4, 1, 2, 7\}$: $\boxed{a = b = c = d = 0}$