Homework

1. Find the characteristics for

a) $$x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2} = 0$$

b) $$\frac{\partial u}{\partial x_1} + u \frac{\partial u}{\partial x_2} = 0$$

2. Find the solution of

$$x_1 x_2 \frac{\partial u}{\partial x_1} - x_2^2 \frac{\partial u}{\partial x_2} = x_1$$

which satisfies : $x_1 = a$, $2 a x_2 u = a^2 + 2$

Homework solutions

(1a) $x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2} = 0$

$$\frac{dx_1}{x_2} = \frac{dx_2}{-x_1} = \frac{du}{0} \Rightarrow du = 0 \Rightarrow \boxed{u = c_2}$$

$$\frac{dx_1}{x_2} = \frac{dx_2}{-x_1} \Rightarrow x_1 dx_1 + x_2 dx_2 = 0 \Rightarrow \boxed{x_1^2 + x_2^2 = c_1^2}$$

Characteristics are circles on which the solution is constant. Parametrized:

$$\begin{cases} x_1 = c_1 \sin(t - c_3) \\ x_2 = c_1 \cos(t - c_3) \\ u = c_2 \end{cases}$$

alternative solution:

$$\left.\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -x_1 \end{cases}\right\} \Rightarrow \frac{dX}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot X = A \cdot X$$

$$\frac{du}{dt} = 0 \Rightarrow u = c_2$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$$

$$\begin{aligned} x_1 &= d_1 \cos t + d_2 \sin t \\ x_2 &= d_3 \cos t + d_4 \sin t \end{aligned} \Rightarrow \begin{aligned} \dot{x}_1 &= -d_1 \sin t + d_2 \cos t \\ \dot{x}_2 &= -d_3 \sin t + d_4 \cos t \end{aligned}$$

$$\left.\begin{aligned} -d_1 \sin t + d_2 \cos t &= d_3 \cos t + d_4 \sin t \\ -d_3 \sin t + d_4 \cos t &= -d_1 \cos t - d_2 \sin t \end{aligned}\right\} \Rightarrow \begin{aligned} -d_1 &= d_4 \\ d_2 &= d_3 \\ -d_3 &= -d_2 \\ d_4 &= -d_1 \end{aligned}$$

$d_2 = d_3$, $d_4 = -d_1$

$$\begin{cases} x_1 = d_1 \cos t + d_3 \sin t \\ x_2 = d_3 \cos t - d_1 \sin t \\ u = c_2 \end{cases} \qquad \left(x_1^2 + x_2^2 = \dots = d_1 + d_3 = c_1^2 \right)$$

Homework solutions

(16) $\frac{\partial u}{\partial x_1} + u \cdot \frac{\partial u}{\partial x_2} = 0$

$$\begin{cases} \frac{dx_1}{dt} = 1 \\ \frac{dx_2}{dt} = u \\ \frac{du}{dt} = 0 \end{cases} \Rightarrow \begin{cases} x_1 = t + C_1 \\ u = C_2 \\ x_2 = C_2 t + C_3 \end{cases}$$

If we seek the solution which for $x_1 = 0$ becomes $u = u_0(x_2)$, that is, to contain the point $(0, x_2, u_0(x_2))$, parametrized $x_1 = 0$, $x_2 = \tau$, $u = u_0(\tau)$, then

$$\begin{matrix} x_2 = \tau \\ x_1 = 0 \\ u = u_0(\tau) \end{matrix} \Rightarrow \begin{matrix} 0 = 0 + C_1 \\ u_0(\tau) = C_2 \\ \tau = C_2 \cdot 0 + C_3 \end{matrix} \Rightarrow \begin{matrix} C_1 = 0 \\ C_2 = u_0(\tau) \\ C_3 = \tau \end{matrix} \Rightarrow \begin{cases} x_1 = t \\ x_2 = u_0(\tau) \cdot t + \tau \\ u = u_0(\tau) \end{cases}$$

is the characteristic which for $t = 0$ passes througth the point $(0, \tau, u_0(\tau))$.

alt. sol.

$$\frac{dx_1}{1} = \frac{dx_2}{u} = \frac{du}{0}$$

$$\Rightarrow du = 0 \Rightarrow \boxed{u_0 = C_1}$$

$$\frac{dx_1}{1} = \frac{dx_2}{C_1} \Rightarrow C_1 x_1 - x_2 = C_2$$

$$\Rightarrow \boxed{u x_1 - x_2 = C_2}$$

Characteristics are intersection of these two surfaces.

Homework solutions

(2) $x_1 x_2 \frac{\partial u}{\partial x_1} - x_2^2 \frac{\partial u}{\partial x_2} = x_1$, $x_1 = a$, $2a x_2 u = a^2 + 2$

$$\frac{dx_1}{x_1 x_2} = \frac{dx_2}{-x_2^2} = \frac{du}{x_1}$$

$$\frac{dx_1}{x_1 \cancel{x_2}} = \frac{dx_2}{-x_2^{\cancel{2}}} \Rightarrow \frac{dx_1}{x_1} + \frac{dx_2}{x_2} = 0 \Rightarrow \boxed{x_1 x_2 = C_1}$$

$$\frac{dx_1}{x_1 x_2} = \frac{du}{x_1} \Rightarrow \frac{dx_1}{C_1} = \frac{du}{x_1} \Rightarrow x_1 dx_1 - C_1 du = 0$$

$$\Rightarrow x_1^2 - 2C_1 u = C_2$$

$$\Rightarrow \boxed{x_1^2 - 2x_1 x_2 u = C_2}$$

General solution:

$F(x_1 x_2 , x_1^2 - 2x_1 x_2 u) = 0$

Cauchy solution:

$$\begin{cases} x_1 x_2 = C_1 \\ x_1^2 - 2x_1 x_2 u = C_2 \\ x_1 = a \\ 2a x_2 u = a^2 + 2 \end{cases} \Rightarrow$$

$a x_2 = C_1$

$a^2 - 2a x_2 u = C_2$

$\left.\begin{array}{l} a^2 - 2C_1 u = C_2 \\ 2C_1 u = a^2 + 2 \end{array}\right\} +$

$a^2 = a^2 + C_2 + 2$

$\Rightarrow C_2 = -2$

$$\Rightarrow \boxed{x_1^2 - 2x_1 x_2 u = -2}$$

or $$u = \frac{x_1^2 + 2}{2x_1 x_2}$$