About systems of differential equations (ODE)

Normal form of ODE system:

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad , \quad i = 1, 2, \dots, n \qquad (1)$$

General solution is

$$x_i(t) = \varphi_i(t, c_1, \dots, c_n) \quad , \quad i = 1, \dots, n$$

Cauchy problem: find the solution of (1) that satisfies

$$x_i(t_0) = a_i \quad , \quad i = 1, \dots, n$$

Each solution is a trajectory in $\mathbb{R}^n$ space. Cauchy solution is a solution which goes through the point $(a_1, \dots, a_n)$.

Symmetrical form of ODE system:

$$\frac{dx_1}{X_1(x_1, \dots, x_{n+1})} = \frac{dx_2}{X_2(x_1, \dots, x_{n+1})} = \dots = \frac{dx_{n+1}}{X_{n+1}(x_1, \dots, x_{n+1})} \qquad (2)$$

We choose one variable to be independent - x_{n+1}

The solution is then

$$x_i = \varphi_i(x_{n+1}, c_1, \dots, c_n) \quad , \quad i = 1, \dots, n$$

Solving (2) we can get relations

$$\psi_i(x_1, \dots, x_{n+1}) = c_i \quad , \quad i = 1, \dots, n$$

When $x_1, \dots, x_{n+1}$ are solutions of (2) each function $\psi_i(x_1, \dots, x_{n+1})$ ~~is~~ reduces to a constant.

Function $\varphi_i(x_1, \ldots, x_{n+1})$ with such feature (to reduce to a constant on the solution) we call the <u>first integral</u> of (2). From the definition, every function $F(\varphi_1, \ldots, \varphi_n)$ is also the first integral.

Solving (2) is equivalent to finding n linearly independent first integrals

$$\varphi_i(x_1, \ldots, x_{n+1}) = C_i \quad , \quad i = 1, \ldots, n .$$

<u>Example</u>: Solve $\dfrac{dx_1}{x_2 x_3} = \dfrac{dx_2}{x_1 x_3} = \dfrac{dx_3}{-2x_1 x_2}$

$$\frac{dx_1}{x_2 \cancel{x_3}} = \frac{dx_2}{x_1 \cancel{x_3}} \Rightarrow x_1 dx_1 = x_2 dx_2 \Rightarrow \frac{1}{2} x_1^2 = \frac{1}{2} x_2^2 + C_1 \cdot \frac{1}{2}$$

$$\boxed{x_1^2 - x_2^2 = C_1}$$

$$\frac{dx_1}{\cancel{x_2} x_3} = \frac{dx_3}{-2x_1 \cancel{x_2}} \Rightarrow -2x_1 dx_1 = x_3 dx_3$$

$$\Rightarrow \boxed{x_1^2 + \frac{1}{2} x_3^2 = C_2}$$

First order PDEs

$u = u(x_1, \dots, x_n)$

$X = (x_1, \dots, x_n)$

$u = u(X)$

Def First order PDE is

$$F\left(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right) = 0$$

or

$$F(X, u, \operatorname{grad} u) = 0 \qquad (3)$$

where the function F is defined in $D \subset \mathbb{R}^{2n+1} = \mathbb{R}^n_X \times \mathbb{R}_u \times \mathbb{R}^n_{\operatorname{grad} u}$

Def Function $u = \varphi(X)$, $X \in Q \subset \mathbb{R}^n_X$ is a solution of (3) if

1) $\varphi \in C^1(Q)$
2) $\forall X \in Q \quad (X, \varphi(X), \operatorname{grad} \varphi(X)) \in D$
3) $\forall X \in Q \quad F(X, \varphi(X), \operatorname{grad} \varphi(X)) \equiv 0$.

The surface $u = \varphi(X)$ is called <u>integral surface</u> of (3)

Quasilinear PDE

$$\sum_{i=1}^{n} A_i(X, u) \frac{\partial u}{\partial x_i} = A_{n+1}(X, u)$$

Linear PDE

$$\sum_{i=1}^{n} A_i(X) \frac{\partial u}{\partial x_i} \pm A_{n+1}(X) \cdot u = A_{n+2}(X)$$

Linear homogeneous PDE

$$\sum_{i=1}^{n} A_i(X) \frac{\partial u}{\partial x_i} = 0$$

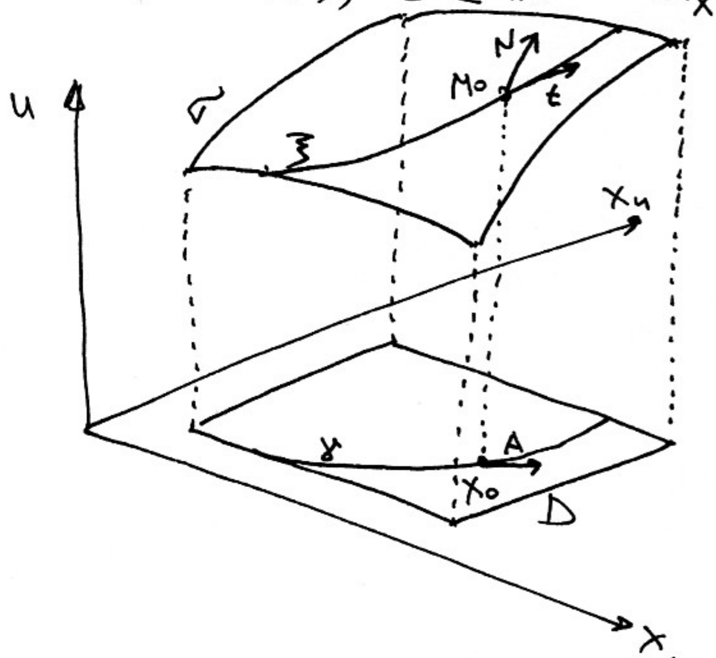
Cauchy problem Find the solution of (3) which contains ~~the~~ some ~~curve~~ $(n-1)$ dimensional hypersurface.

Method of characteristics for 1st order PDE

Quasilinear PDE

$$\sum_{i=1}^{n} A_i(X,u)\frac{\partial u}{\partial x_i} = A_{n+1}(X,u) \qquad (1)$$

$$\left(A_i \in C^1(G),\ G \subset \mathbb{R}^{n+1} = \mathbb{R}^n_X \times \mathbb{R}_u,\ \sum_1^n A_i^2 \neq 0 \right)$$



$D \subset \mathbb{R}^n_X$

$\sigma : u = u(X)$

$\gamma : X = \varphi(t)$

$\xi : \xi(t) = u(\varphi(t))$

Let us choose the curve γ so that the tangent vector $\frac{dX}{dt}$ is egual (at every point of γ) to

$$A = \left(A_1(\varphi(t), \xi(t)), \dots, A_n(\varphi(t), \xi(t))\right)$$

that means, to be

$$\frac{dx_i}{dt} = A_i(\varphi(t), \xi(t)) \quad , \ i = 1, \dots, n$$

Differentiation of $\xi(t) = u(\varphi(t))$ w.r.t. t yealds

$$\frac{d\xi}{dt} = \sum_{i=1}^{n} \frac{\partial u(\varphi(t))}{\partial x_i} \cdot \frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial u(\varphi(t))}{\partial x_i} \cdot A_i(\varphi(t), \xi(t)) = A_{n+1}(\varphi(t), \xi(t))$$

From here it follows that the curve ξ such that $X = \varphi(t)$, $u = \xi(t)$ is a trajectory of the ODE system

$$\begin{cases} \dfrac{dx_i}{dt} = A_i(X,u) \quad , \ i = 1, \dots, n \\[2mm] \dfrac{du}{dt} = A_{n+1}(X,u) \end{cases} \qquad (2)$$

<u>Def</u> The system (2) is called a <u>system for characteristics ~~trajectories~~</u> for PDE (1). Phase trajectories of ODE system (2) are called <u>characteristics</u> of PDE equation (1).

$\boxed{T}$ If $M_0 = (X_0, u_0) \in \sigma : u = u(X)$ is a solution of (1) then σ contains the characteristic of (1) which goes through M_0. (Integral surface is made of characteristics).

<u>General solution of quasilinear PDE</u>

For eq. (1) we write the associate system

$$\frac{dx_1}{A_1} = \frac{dx_2}{A_2} = \dots = \frac{dx_n}{A_n} = \frac{du}{A_{n+1}}$$

and find n independent first integrals

$$v_1(X,u) = C_1, \dots, v_n(X,u) = C_n .$$

General solution of (1) is given by the equation

$$F(v_1(X,u), \dots, v_n(X,u)) = 0 .$$

<u>Example</u>

Find the characteristics for $a \cdot \frac{\partial u}{\partial x_1} + b \frac{\partial u}{\partial x_2} = c$.

S: System for characteristics is

$$\frac{dx_1}{dt} = a \ ; \ \frac{dx_2}{dt} = b \ ; \ \frac{du}{dt} = c$$

and

$$\begin{cases} x_1 = at + x_1^0 \\ x_2 = bt + x_2^0 \\ u = ct + u^0 \end{cases}$$

Example Find the surface which satisfies PDE

$$-x_1 \frac{\partial u}{\partial x_1} + \frac{u}{2x_2}\frac{\partial u}{\partial x_2} = u \qquad (x_2 > 0)$$

and contains the curve (line) $x_1 + x_2 = 1$, $u = x_2$

Sol.

Curve (line) γ is parametrized as: $x_1 = 1-\tau$
$x_2 = \tau$
$u = \tau$

System for characteristics is

$$\begin{cases} \frac{dx_1}{dt} = -x_1 \\ \frac{dx_2}{dt} = \frac{u}{2x_2} \\ \frac{du}{dt} = u \end{cases} \Rightarrow \begin{cases} x_1 = c_1 e^{-t} \\ x_2 = \sqrt{c_3 e^t + c_3} \\ u = c_3 e^t \end{cases}$$

phase trajectories

The solution (integral surface) σ is made of characteristics which for $t=0$ intersect the curve γ. It follows:

$$x_1 = 1-\tau = c_1 \cdot e^{-0} = c_1$$
$$x_2 = \tau = \sqrt{c_3 e^0 + c_2} = \sqrt{c_2 + c_3}$$
$$u = \tau = c_3 e^0 = c_3$$

wherefrom

$$c_1 = 1-\tau, \quad c_2 = \tau^2 - \tau, \quad c_3 = \tau$$

and, finally

$$\begin{cases} x_1 = (1-\tau)e^{-t} \\ x_2 = \sqrt{\tau e^t + \tau^2 - \tau} \\ u = \tau e^t \end{cases}$$

Eliminating t and τ from above, we can get

$$\sigma: \quad u = \frac{x_2^2}{1-x_1}$$

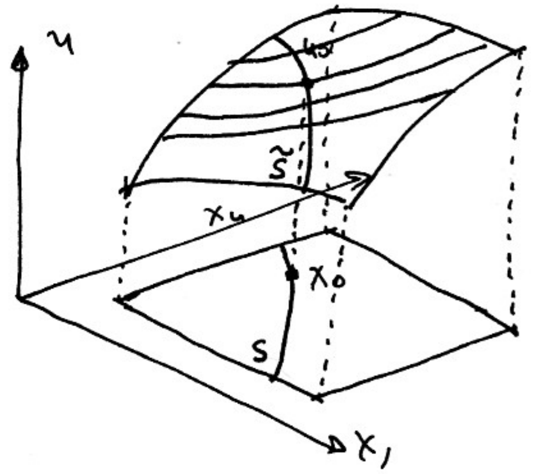
Cauchy problem for quasilinear PDE

$$\sum_{i=1}^{n} A_i(X,u)\frac{\partial u}{\partial x_i} = A_{n+1}(X,u) \qquad (1)$$

smooth hypersurface $S: X = \Psi(t_1, t_2, \dots, t_{n-1})$, $(t_1,\dots,t_{n-1}) \in T$.

Find a solution of (1) for which

$$u\big|_S = u_0(t_1, t_2, \dots, t_{n-1}) \qquad (2).$$



$\tilde{S}: X = \Psi(t_1,\dots,t_{n-1}),\ u = u_0(t_1,\dots,t_{n-1})$

$\boxed{n=2}$ Find a solution $u = u(x_1, x_2)$ which contains a curve $X = \Psi(t)$, $u = u_0(t)$.

The solution consists of all characteristics from points of hypersurface $\tilde{S}$.

$\div$

Using n first integrals of (1), from

$v_1(X,u) = C_1, \dots, v_n(X,u) = C_n,$

$X = \Psi(t), \quad u = u_0(t)$

we eliminate all the variables to get the relation between constants

$$F(C_1, C_2, \dots, C_n) = 0.$$

The Cauchy solution is then given by

$$F(v_1(X,u), \dots, v_n(X,u)) = 0$$

<u>Example</u> Find the Cauchy solution of

$$x_2 u \frac{\partial u}{\partial x_1} + x_1 u \frac{\partial u}{\partial x_2} = -2x_1 x_2 \quad , \quad x_2 = 0 , \; u = x_1$$

Solution: from previous example, first integrals are

$$x_1^2 - x_2^2 = c_1$$
$$\frac{1}{2}u^2 + x_1^2 = c_2$$

and general solution is

$$F\left(x_1^2 - x_2^2 , \; \frac{1}{2}u^2 + x_1^2\right) = 0$$

or

$$\frac{1}{2}u^2 + x_1^2 = f\left(x_1^2 - x_2^2\right)$$
$$u^2 = 2\left(f(x_1^2 - x_2^2) - x_1^2\right) .$$

Cauchy solution

$$\left.\begin{array}{l} x_1^2 - x_2^2 = c_1 \\ \frac{1}{2}u^2 + x_1^2 = c_2 \\ x_2 = 0 \\ u = x_1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_1^2 = c_1 \\ \frac{1}{2}x_1^2 + x_1^2 = c_2 \end{array}\right\} \Rightarrow \frac{3}{2}c_1 = c_2$$

$$3c_1 = 2c_2$$
$$3\left(x_1^2 - x_2^2\right) = 2\left(\frac{1}{2}u^2 + x_1^2\right)$$
$$u^2 = x_1^2 - 3x_2^2 .$$