Задатак 0.1. Израчунати $\int_0^1 \frac{\mathrm{d}x}{\sqrt{2x^2+4x+3}}$.

Решење.

$$\begin{aligned} I &= \int_0^1 \frac{\mathrm{d}x}{\sqrt{2x^2+4x+3}} = \frac{1}{\sqrt{2}}\int_0^1 \frac{\mathrm{d}x}{\sqrt{x^2+2x+\frac{3}{2}}} = \frac{1}{\sqrt{2}}\int_0^1 \frac{\mathrm{d}x}{\sqrt{(x+1)^2+\frac{1}{2}}} \\ &= \begin{pmatrix} x+1=t \\ \mathrm{d}x=\mathrm{d}t \end{pmatrix} = \frac{1}{\sqrt{2}}\int_1^2 \frac{\mathrm{d}t}{\sqrt{t^2+\frac{1}{2}}} = \frac{1}{\sqrt{2}}\ln\left(t+\sqrt{t^2+\frac{1}{2}}\right)\Bigg|_1^2 \\ &= \frac{1}{\sqrt{2}}\ln\frac{2+\sqrt{\frac{9}{2}}}{1+\sqrt{\frac{3}{2}}} = \frac{1}{\sqrt{2}}\ln\frac{2\sqrt{2}+3}{\sqrt{2}+\sqrt{3}} = \frac{1}{\sqrt{2}}\ln\left(2\sqrt{6}-4+3\sqrt{3}-3\sqrt{2}\right). \end{aligned}$$

△

Задатак 0.2. Израчунати $I = \int_2^3 \frac{\mathrm{d}x}{x^3+x^2-2}$.

Решење.

$$\int_2^3 \frac{\mathrm{d}x}{x^3+x^2-2} = \int_2^3 \frac{\mathrm{d}x}{(x-1)(x^2+2x+2)},$$

Множењем једнакости

$$\frac{1}{x^3+x^2-2} = \frac{1}{(x-1)(x^2+2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}$$

са $(x+1)(x^2+2x+2)$ добијамо систем

$$A+B=o,\ 2A-B+C=0,\ 2A-C=1$$

чије је решење $A=\frac{1}{5}$, $B=-\frac{1}{5}$, $C=-\frac{3}{5}$. Према томе, интеграл је једнак

$$\int_2^3 \left(\frac{\frac{1}{5}}{x-1} + \frac{-\frac{1}{5}x-\frac{3}{5}}{x^2+2x+2}\right) = \frac{1}{5}\int_2^3 \frac{\mathrm{d}x}{x-1} - \frac{1}{5}\int_2^3 \frac{x+3}{x^2+2x+2}\,\mathrm{d}x = \frac{1}{5}I_1 - \frac{1}{5}I_2,$$

$$I_1 = \int_2^3 \frac{\mathrm{d}x}{x-1} = \ln(x-1)|_2^3 = \ln 2,$$

$$\begin{aligned} I_2 &= \int_2^3 \frac{x+3}{x^2+2x+2}\,\mathrm{d}x = \frac{1}{2}\int_2^3 \frac{2x+6}{x^2+2x+2}\,\mathrm{d}x = \frac{1}{2}\int_2^3 \frac{2x+2}{x^2+2x+2}\,\mathrm{d}x + \frac{1}{2}\int_2^3 \frac{4}{x^2+2x+2}\,\mathrm{d}x \\ &= \frac{1}{2}\ln(x^2+2x+2)|_2^3 + 2\int_2^3 \frac{\mathrm{d}x}{(x+1)^2+1} = \frac{1}{2}\ln\frac{17}{10} + 2\mathrm{arctg}(x+1)|_2^3 \\ &= \frac{1}{2}\ln\frac{17}{10} + 2\mathrm{arctg}\,4 - 2\mathrm{arctg}\,3. \end{aligned}$$

Добили смо

$$I = \frac{1}{5}I_1 - \frac{1}{5}I_2 = \frac{1}{5}\ln 2 - \frac{1}{10}\ln\frac{17}{10} - \frac{2}{5}\text{arctg}\,4 + \frac{2}{5}\text{arctg}\,3.$$

△

Задатак 0.3. Израчунати $I = \int x^2 \text{arctg}\,\frac{1}{x}\,dx$.

Решење.

$$\begin{aligned} I &= \int x^2\text{arctg}\,\frac{1}{x}\,dx = \begin{pmatrix} u = \text{arctg}\,\frac{1}{x} \;\Rightarrow\; \mathrm{d}u = \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)\mathrm{d}x = -\frac{\mathrm{d}x}{x^2+1} \\ dv = x^2\,\mathrm{d}x \;\Rightarrow\; v = \frac{x^3}{3} \end{pmatrix} \\ &= \frac{x^3}{3}\cdot\text{arctg}\,\frac{1}{x} + \int\frac{x^3}{3}\frac{\mathrm{d}x}{x^2+1} = \frac{x^3}{3}\cdot\text{arctg}\,\frac{1}{x} + \frac{1}{3}\int\frac{x^3\mathrm{d}x}{x^2+1} = \frac{x^3}{3}\cdot\text{arctg}\,\frac{1}{x} + \frac{1}{3}I_1, \end{aligned}$$

$$\begin{aligned} I_1 &= \int\frac{x^3\mathrm{d}x}{x^2+1} = \begin{pmatrix} x^2+1=t \\ 2x\mathrm{d}x = \mathrm{d}t \end{pmatrix} = \frac{1}{2}\int\frac{(t-1)\mathrm{d}t}{t} = \frac{1}{2}(t - \ln|t|) + C \\ &= \frac{1}{2}\left(x^2+1-\ln|x^2+1|\right) + C, \end{aligned}$$

$$I = \frac{1}{3}x^3\text{arctg}\,\frac{1}{x} + \frac{1}{6}(x^2+1) - \frac{1}{6}\ln(x^2+1) + C.$$

△

Задатак 0.4. Израчунати $\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{13 + \text{arctg}\,x + e^x\cos^2 x}{\cos x}\,dx$.

Решење. Важи

$$\begin{aligned} I &= \int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{13+\text{arctg}\,x + e^x\cos^2 x}{\cos x}\,dx = \int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{13}{\cos x} + \frac{\text{arctg}\,x}{\cos x} + e^x\cos x\right)dx \\ &= 13\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\mathrm{d}x}{\cos x} + \int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\text{arctg}\,x}{\cos x}\,\mathrm{d}x + \int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^x\cos x\,dx = I_1 + I_2 + I_3, \end{aligned}$$

при чему је

$$\begin{aligned} I_1 &= 13\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\mathrm{d}x}{\cos x} = 13\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos x\mathrm{d}x}{\cos^2 x} = \begin{pmatrix} \sin x = t \\ \cos x\mathrm{d}x = \mathrm{d}t \end{pmatrix} = 13\int\limits_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\frac{\mathrm{d}t}{1-t^2} = \frac{13}{2}\ln\frac{1+x}{1-x}\Bigg|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \\ &= \frac{13}{2}\left(\ln\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} - \ln\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\right) = \frac{13}{2}\left(\ln\frac{2+\sqrt{2}}{2-\sqrt{2}} - \ln\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) = 13\ln\frac{2+\sqrt{2}}{2-\sqrt{2}}, \end{aligned}$$

$$I_2 = \int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\operatorname{arctg} x}{\cos x}\,\mathrm{d}x = 0.$$

Урадимо неодређени интеграл

$$\begin{aligned} I_3 &= \int e^x \cos x \mathrm{d}x = \begin{pmatrix} u = \cos x & \Rightarrow & \mathrm{d}u = -\sin x \mathrm{d}x \\ dv = e^x & \Rightarrow & v = e^x \end{pmatrix} \\ &= e^x \cos x + \int e^x \sin x \mathrm{d}x = \begin{pmatrix} u = \sin x & \Rightarrow & \mathrm{d}u = \cos x \\ dv = e^x & \Rightarrow & v = e^x \end{pmatrix} \\ &= e^x \cos x + \left(e^x \sin x - \int \cos x e^x \right) \\ &= e^x \cos x + e^x \sin x - \int \cos x e^x = e^x \cos x + e^x \sin x - I_3. \end{aligned}$$

Добили смо да је

$$I_3 = \frac{1}{2} e^x (\cos x + \sin x) + C,$$

одакле је

$$I_3 = \frac{1}{2} e^x (\cos x + \sin x) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2} e^{\frac{\pi}{4}} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}.$$

Према томе,

$$I = 13 \ln \frac{2+\sqrt{2}}{2-\sqrt{2}} + \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}.$$

△

Задатак 0.5. Израчунати обим и површину фигуре ограничене кривом $(x-1)^2+y^2=4$ и правама $y=0$, $x=1$ и $\sqrt{3}x+3y-3\sqrt{3}=0$. Затим, одредити запремину тела које се добија ротацијом дате фигуре око x-осе.

Решење.

$$\begin{aligned} P &= \int\limits_{-1}^{0} \sqrt{4-(x-1)^2}\,\mathrm{d}x + \frac{2\cdot\sqrt{3}}{2} = \int\limits_{-1}^{0} \sqrt{4-(x-1)^2}\,\mathrm{d}x + \sqrt{3} = \begin{pmatrix} x-1=t \\ \mathrm{d}x = \mathrm{d}t \end{pmatrix} \\ &= \int\limits_{-2}^{-1} \sqrt{4-t^2}\,\mathrm{d}t + \sqrt{3} = \begin{pmatrix} t = 2\sin u \\ \mathrm{d}t = 2\cos u \mathrm{d}u \end{pmatrix} = 4 \int\limits_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} \cos^2 u \mathrm{d}u + \sqrt{3} \\ &= 2 \int\limits_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} (1+\cos 2u)\mathrm{d}u + \sqrt{3} = 2\left(u + \frac{1}{2}\sin 2u \right)\Big|_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} + \sqrt{3} = (2u+\sin 2u)|_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} + \sqrt{3} \\ &= 2\left(-\frac{\pi}{6}+\frac{\pi}{2}\right) - \frac{1}{2} + \sqrt{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} + \sqrt{3} = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}. \end{aligned}$$

$$\begin{aligned} O &= O_1 + O_2 + O_3 + O_4 = 2 + \frac{2\sqrt{3}}{3} + \sqrt{(0-1)^2 + \left(\sqrt{3} - \frac{2\sqrt{3}}{3} \right)^2} + O_4 \\ &= 2 + \frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{3} + O_4 = 2 + \frac{4\sqrt{3}}{3} + O_4, \end{aligned}$$

$$
\begin{aligned}
O_4 &= \int_{-1}^{0} \sqrt{1+[y'(x)]^2}\,\mathrm{d}x = \int_{-1}^{0} \sqrt{1+\left[\left(\sqrt{4-(x-1)^2}\right)'\right]^2}\,\mathrm{d}x \\
&= \int_{-1}^{0} \sqrt{1+\left[\frac{-2(x-1)}{2\sqrt{4-(x-1)^2}}\right]^2}\,\mathrm{d}x = \int_{-1}^{0} \sqrt{1+\frac{(x^2-2x+1)}{4-(x-1)^2}}\,\mathrm{d}x \\
&= \int_{-1}^{0} \sqrt{\frac{4-x^2+2x-1+x^2-2x+1}{4-(x-1)^2}}\,\mathrm{d}x = 2\int_{-1}^{0} \frac{1}{\sqrt{4-(x-1)^2}}\,\mathrm{d}x \begin{pmatrix} t=x-1 \\ \mathrm{d}t=\mathrm{d}x \end{pmatrix} \\
&= 2\int_{-2}^{-1} \frac{1}{\sqrt{4-t^2}}\,\mathrm{d}t = 2\arcsin\frac{t}{2}\Big|_{-2}^{-1} = 2\left(-\frac{\pi}{6}+\frac{\pi}{2}\right) = \frac{2\pi}{3},
\end{aligned}
$$

$$
O = 2+\frac{4\sqrt{3}}{3}+O_4 = 2+\frac{4\sqrt{3}}{3}+\frac{2\pi}{3}.
$$

$$
\begin{aligned}
V &= V_1+V_2 = \pi\int_{-1}^{0}\left(\sqrt{4-(x-1)^2}\right)^2\mathrm{d}x + \pi\int_{0}^{1}\left(\sqrt{3}-\frac{\sqrt{3}}{3}x\right)^2\mathrm{d}x \\
&= \pi\int_{-1}^{0}(3+2x-x^2)\,\mathrm{d}x + \pi\int_{0}^{1}\left(3-2x+\frac{1}{3}x^2\right)\,\mathrm{d}x \\
&= \pi\left(3x+x^2-\frac{x^3}{3}\right)\Big|_{-1}^{0} + \pi\left(3x-x^2-\frac{x^3}{9}\right)\Big|_{0}^{1} = \pi\left(3-1-\frac{1}{3}\right)+\pi\left(3-1-\frac{1}{9}\right) \\
&= \frac{34}{9}\,\pi.
\end{aligned}
$$

△