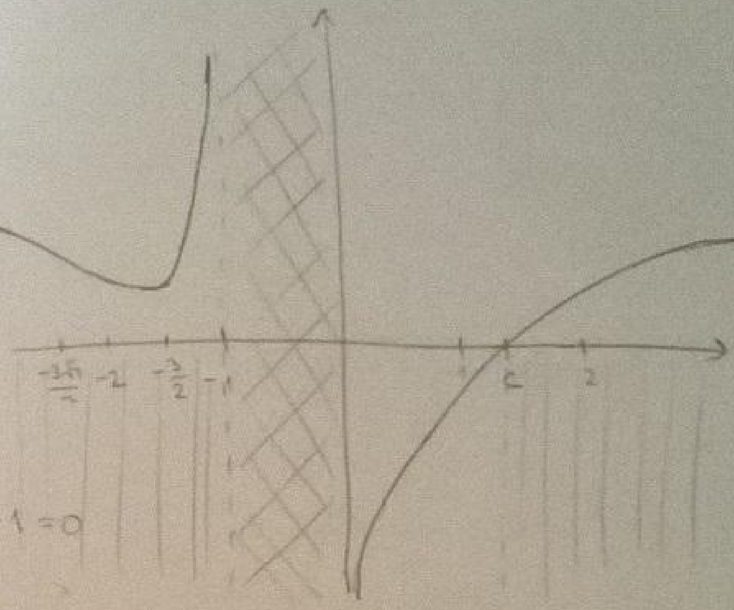
$\boxed{1}$ 1) $f(x) = \ln \frac{x^3}{x+1}$



• <u>Df</u> $= (-\infty, -1) \cup (0, +\infty)$

$\frac{x^3}{x+1} > 0 \leftrightarrow$ $+$ -1 $-$ 0 $+$

• <u>Parnost / periodičnost</u>: × (obrazloženi...)

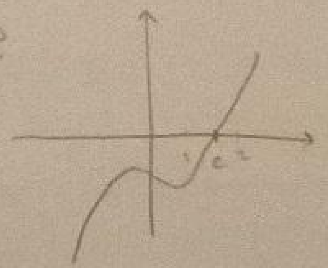
• <u>Nule i znak</u>:

$f(x) = 0 \Leftrightarrow \frac{x^3}{x+1} = 1 \Leftrightarrow x^3 - x - 1 = 0$

$g(x) = x^3 - x - 1$

$\left.\begin{matrix} g(1) = -1 \\ g(2) = 5 \end{matrix}\right\} \overset{B-K}{\Rightarrow} (\exists c \in (1,2))\ g(c) = 0$

Da li ima još realnih nula?
Nema!



$\lim_{x\to+\infty} g(x) = +\infty$

$\lim_{x\to-\infty} g(x) = -\infty$

$g'(x) = 3x^2 - 1 = (\sqrt{3}x - 1)(\sqrt{3}x + 1)$

$f(x) > 0 \Leftrightarrow \frac{x^3}{x+1} > 1 \Leftrightarrow \frac{x^3 - x - 1}{x+1} > 0$

$\Leftrightarrow x < -1 \lor x > c$

• <u>Asimptote</u>

• VA $\lim_{x\to-1_-} f(x) = \ln\Big(\lim_{x\to-1_-} \underbrace{\frac{x^3}{x+1}}_{\nearrow +\infty}\Big) = +\infty \Rightarrow x = -1$ je V.A sa leve strane

$\lim_{x\to 0_+} f(x) = \ln\Big(\lim_{x\to 0_+} \underbrace{\frac{x^3}{x+1}}_{\downarrow 0}\Big) = -\infty \Rightarrow x = 0$ je V.A sa desne strane

• HA $\left.\begin{matrix} \lim_{x\to+\infty} f(x) = \text{„}\ln +\infty\text{"} = +\infty \\ \lim_{x\to-\infty} f(x) = \text{„}\ln +\infty\text{"} = +\infty \end{matrix}\right\}$ nema H.A.

• KA $k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\ln \frac{x^3}{x+1}}{x} \overset{\frac{\infty}{\infty}}{\underset{L}{=}} \lim_{x\to\pm\infty} \frac{x+1}{x^3} \cdot \frac{3x^2(x+1) - x^3}{(x+1)^2}$

$= \lim_{x\to\pm\infty} \frac{2x^3 + 3x^2}{x^{\cancel{3}}(x+1)} = \lim_{x\to\pm\infty} \frac{3 + 2x}{x(x+1)} = 0 \Rightarrow$ nema kose

• <u>Monotonost</u> $f'(x) = \frac{x+1}{x^3} \cdot \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} = \frac{2x+3}{x(x+1)}$

$f'(x)=0 \iff x=-\frac{3}{2}$

$f'(x)>0 \iff x\in(-\frac{3}{2},-1)$

$\qquad\qquad\quad x\in(0,+\infty)$

$f'(x)<0 \iff x\in(-\infty,-\frac{3}{2})$

Tačka $(-\frac{3}{2}, \ln\frac{27}{4})$ je tačka lokalnog minimuma

$\left[\, f(-\frac{3}{2}) = \ln\frac{-\frac{27}{8}}{-\frac{1}{2}} = \ln\frac{27}{4} \,\right]$

• **Konveksnost**

$$f''(x) = \left(\frac{2x+3}{x(x+1)}\right)' = \frac{2(x^2+x)-(2x+3)(2x+1)}{x^2(x+1)^2} = \frac{2x^2+2x-4x^2-8x-3}{x^2(x+1)^2}$$

$$= \frac{-2x^2-6x-3}{x^2(x+1)^2} = -\frac{2x^2+6x+3}{x^2(x+1)^2}$$

$$x_{1,2} = \frac{-6\pm\sqrt{36-24}}{4} = \frac{-6\pm2\sqrt{3}}{4} = \frac{-3\pm\sqrt{3}}{2}$$

Tačka $(\frac{-3-\sqrt{3}}{2}, f(\frac{-3-\sqrt{3}}{2}))$ je prevojna tačka.

b) $f(x) = \frac{2x+3}{x(x+1)}$

$\left.\begin{array}{l} f\in C[2,+\infty) \\ \lim\limits_{x\to+\infty} f(x)=0 \end{array}\right\}$ $f(x)$ je ograničena f-ja na $(2,+\infty)$

$\Rightarrow f$ je ravnomerno neprekidna na $(2,+\infty)$

c) F-ja $f(x)$ nije ograničena u okolini tačke $x=0$, pa ne može biti ravnomerno neprekidna na skupu $(0,1]$.

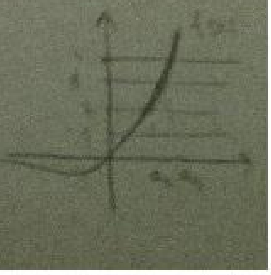
$\boxed{2}$ $\quad a_n e^{a_n} = n, \quad n\geq 1$

a) Posmatrajmo f-ju $f(x)=x e^x$. Ako pokažemo da je f bijekcija na $(0,+\infty)$ tada je f^{-1}: $a_n = f^{-1}(n)$, $n\geq1$.

$f(x)=x e^x$

$f'(x) = e^x + x e^x = \underbrace{e^x}_{>0}\underbrace{(x+1)}_{>0} > 0, \quad x\in(0,+\infty)$

$f \nearrow$ i neprekidna je $\Rightarrow f$ je bijekcija na $(0,+\infty)$



Odatle za svako $n\in\mathbb{N}$ postoji jedinstveno a_n t.d. je

$a_n e^{a_n} = n$.

b) $a_n \underbrace{e^{a_n}}_{>0} = \underbrace{n}_{>0} \Rightarrow a_n > 0$ za svaki $n \in \mathbb{N}$

c) Pretpostavimo suprotno; da postoje m i n t.d.

$$n < m \Rightarrow a_n \geq a_m$$

$$e^{a_n} \geq e^{a_m} \quad (e^x \text{ je rastuća f-ja})$$

$$\overset{!}{\Rightarrow} a_n e^{a_n} \geq a_m e^{a_m} \quad (\text{jer su pozitivne pa smo smjeli})$$

$$\overset{\shortparallel}{n} \geq \overset{\shortparallel}{m} \quad ↯$$

d) $(a_n)_{n\in\mathbb{N}} \nearrow \Rightarrow \exists \lim\limits_{n\to\infty} a_n = a \in \overline{\mathbb{R}}$

$$a_n e^{a_n} = n \quad / \lim_{n\to\infty}$$

$$a e^a = +\infty$$

$$\Rightarrow \boxed{a = +\infty}$$

e) $\lim\limits_{x\to+\infty} \frac{\ln x}{x} = 0$ (logaritamska je sporija od stepene)

$\overset{\text{Heine}}{\Longrightarrow} \lim\limits_{n\to\infty} \frac{\ln a_n}{a_n} = 0$ (koristimo da je $\lim\limits_{n\to\infty} a_n = +\infty$)

f) $\lim\limits_{n\to\infty} \frac{a_n}{\ln n} = \frac{1}{1} = 1$

$$a_n e^{a_n} = n \quad / \ln$$

$$(\ln a_n) + a_n = \ln n$$

$$\Rightarrow \lim_{n\to\infty} \frac{\ln n}{a_n} = \lim_{n\to\infty} \frac{a_n + \ln a_n}{a_n} = 1 + 0 = 1$$

$\boxed{3.}$ a) $\underbrace{e^x \sin x = 1}_{(*)} \Leftrightarrow e^x \sin x - 1 = 0 \,/\, e^x \Leftrightarrow \sin x - e^{-x} = 0$ dozvoljeno, jer e^x nema nule

$F(x) = \sin x - e^{-x}$ ← pomoćna f-ja

Neka su c_1 i c_2 rješenja jednačine (*). Tada su $c_1, c_2 \in \mathbb{R}$ nule f-je $F(x)$

$$\left.\begin{array}{l} F(c_1) = F(c_2) = 0 \\ F \in C[c_1,c_2] \cap D(c_1,c_2) \end{array}\right\} \overset{\text{Rol}}{\Longrightarrow} (\exists d_1 \in (c_1,c_2))\, F'(d_1) = 0$$

$$F'(x) = \cos x + e^{-x}$$

$$\cos d_1 + e^{-d_1} = 0 \quad / e^{d_1}$$

$$e^{d_1} \cos d_1 = -1$$

tj. $d_1 \in \mathbb{R}$ je rješenje jednačine $e^x \cos x = -1$ koje se nalazi između c_1 i c_2

b) $e^x \cos x + 1 = 0$

$f(x) = e^x \cos x + 1$

- Ako je $x < 0$, onda je $\left.\begin{array}{l} e^x < 1 \\ |\cos x| \leq 1 \end{array}\right\} \; |e^x \cos x| < 1$

$\Rightarrow f(x) > 0$, tj. $f(x)$ nema nula na $(-\infty, 0)$

- Ako je $x \in [0, \frac{\pi}{4}]$ onda je $\left.\begin{array}{l} e^x \geq 1 \\ \cos x \in [\frac{\sqrt{2}}{2}, 1] \end{array}\right\} \Rightarrow f(x) > 0$ na $[0, \frac{\pi}{4}]$

- $f'(x) = e^x \cos x - e^x \sin x = \underbrace{e^x \cdot \sqrt{2}}_{>0} \cos(x + \frac{\pi}{4})$

(∗) $f'(x) > 0 \iff x \in (-\frac{3\pi}{4} + 2k\pi, \frac{\pi}{4} + 2k\pi)$,

(♡) $f'(x) < 0 \iff x \in (\frac{\pi}{4} + 2k\pi, \frac{5\pi}{4} + 2k\pi)$, $k \in \mathbb{Z}$

- $\left.\begin{array}{l} f(\frac{\pi}{4} + 2k\pi) = \frac{\sqrt{2}}{2} e^{\frac{\pi}{4} + 2k\pi} + 1 > 0 \\ f(\frac{5\pi}{4} + 2k\pi) = -\frac{\sqrt{2}}{2} e^{\frac{5\pi}{4} + 2k\pi} + 1 < 0, \; k \geq 0 \end{array}\right\}$ B-K $\Rightarrow$ postoji bar jedna nula na svakom intervalu oblika $(\frac{\pi}{4} + 2k\pi, \frac{5\pi}{4} + 2k\pi)$, $k \geq 0$; zbog (♡) postoji tačno jedna

- $\left.\begin{array}{l} f(-\frac{3\pi}{4} + 2k\pi) = \frac{\sqrt{2}}{2} e^{-\frac{3\pi}{4} + 2k\pi} + 1 < 0, \; k \geq 1 \\ f(\frac{\pi}{4} + 2k\pi) > 0 \end{array}\right\} \overset{B-K}{\Rightarrow}$ -||-

Kako je $[\frac{\pi}{4}, +\infty) = \bigcup_{k \geq 1} [-\frac{3\pi}{4} + 2k\pi, \frac{\pi}{4} + 2k\pi] \cup \bigcup_{k \geq 0} [\frac{\pi}{4} + 2k\pi, \frac{5\pi}{4} + 2k\pi]$

i kako u svakom od ovih segmenata postoji tačno jedno rešenje jednačine $e^x \cos x + 1 = 0$, rešenja ima prebrojivo mnogo (jer ima prebrojivo mnogo segmenata).