$$\operatorname{tg}\frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

①

$$I = \int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\operatorname{tg} x)\,dx = \left(\begin{array}{l} t = -x+\frac{\pi}{3} \\ dt = -dx \end{array}\right)$$

$$= \int_{\frac{\pi}{3}}^{0} \ln\left(1+\sqrt{3}\operatorname{tg}\left(\frac{\pi}{3}-t\right)\right)(-dt) = \int_0^{\frac{\pi}{3}} \ln\left(1+\sqrt{3}\cdot\frac{\operatorname{tg}\frac{\pi}{3}-\operatorname{tg} t}{1+\operatorname{tg}\frac{\pi}{3}\operatorname{tg} t}\right)dt$$

$$= \int_0^{\frac{\pi}{3}} \ln\left(1+\sqrt{3}\,\frac{\sqrt{3}-\operatorname{tg} t}{1+\sqrt{3}\operatorname{tg} t}\right)dt = \int_0^{\frac{\pi}{3}} \ln\frac{1+\cancel{\sqrt{3}\operatorname{tg} t}+3-\cancel{\sqrt{3}\operatorname{tg} t}}{1+\sqrt{3}\operatorname{tg} t}\,dt$$

$$= \int_0^{\frac{\pi}{3}} \ln\frac{4}{1+\sqrt{3}\operatorname{tg} t}\,dt = \ln 4\cdot\frac{\pi}{3} - \int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\operatorname{tg} t)\,dt$$

$$= \ln 4\cdot\frac{\pi}{3} - I \qquad \ln 4 = 2\ln 2$$

$$2I = \ln 4\cdot\frac{\pi}{3} \implies \boxed{I = \frac{\pi}{3}\ln 2}$$

②

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \left(\begin{array}{ll} x = \operatorname{tg} t & \text{слика } [0,\tfrac{\pi}{4}] \text{ на } [0,1] \\ t = \operatorname{arctg} x & dt = \frac{dx}{1+x^2} \end{array}\right)$$

$$I = \int_0^{\frac{\pi}{4}} \ln(1+\operatorname{tg} t)\,dt = \left(\begin{array}{l} u = -t+\frac{\pi}{4} \\ du = -dt \end{array}\right)$$

$$= \int_{\frac{\pi}{4}}^{0} \ln\left(1+\operatorname{tg}\left(\frac{\pi}{4}-u\right)\right)(-du) = \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\operatorname{tg}\frac{\pi}{4}-\operatorname{tg} u}{1+\operatorname{tg}\frac{\pi}{4}\operatorname{tg} u}\right)du$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\operatorname{tg} u}{1+\operatorname{tg} u}\right)du = \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\cancel{\operatorname{tg} u}+1-\cancel{\operatorname{tg} u}}{1+\operatorname{tg} u}\right)du$$

$$= \ln 2\cdot\frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \ln(1+\operatorname{tg} u)\,du \implies 2I = \ln 2\cdot\frac{\pi}{4}$$

$$\boxed{I = \ln 2\cdot\frac{\pi}{8}}$$

③

$$I = \int_{\frac{\pi}{2}}^{\pi} \frac{\sqrt{1+\sin 2x}}{\sin^4\left(\frac{3\pi}{4}-x\right)+2(1-\sin 2x)}\,dx = \left(\begin{array}{l} t=\frac{3\pi}{4}-x \\ dt=-dx \end{array}\right)$$

$$= \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\sqrt{1+\sin\left(2\left(\frac{3\pi}{4}-t\right)\right)}}{\sin^4 t+2\left(1-\sin\left(2\left(\frac{3\pi}{4}-t\right)\right)\right)}\,(-dt)$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sqrt{1+\sin\left(\frac{3\pi}{2}-2t\right)}}{\sin^4 t+2\cdot\left(1-\sin\left(\frac{3\pi}{2}-2t\right)\right)}\,dt$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sqrt{1+(-\cos 2t)}}{\sin^4 t+2(1+\cos 2t)}\,dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sqrt{1-\cos 2t}}{\sin^4 t+2\cdot 2\cos^2 t}\,dt$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sqrt{2\sin^2 t}}{\sin^4 t+4\cos^2 t}\,dt = 2\cdot\int_{0}^{\frac{\pi}{4}} \frac{\sqrt{2}\,|\sin t|}{\sin^4 t+4\cos^2 t}\,dt$$

$$= 2\sqrt{2}\int_{0}^{\frac{\pi}{4}} \frac{\sin t}{(1-\cos^2 t)^2+4\cos^2 t}\,dt = 2\sqrt{2}\cdot\int_{0}^{\frac{\pi}{4}} \frac{\sin t\,dt}{(1+\cos^2 t)^2} = \left(\begin{array}{l} y=\cos t \\ dy=-\sin t\,dt \end{array}\right)$$

$$= 2\sqrt{2}\int_{1}^{\frac{\sqrt{2}}{2}} \frac{-dy}{(1+y^2)^2} = 2\sqrt{2}\int_{\frac{\sqrt{2}}{2}}^{1} \frac{dy}{(1+y^2)^2}$$

$$\int \frac{dy}{1+y^2} = \operatorname{arctg} y + C = \left(\begin{array}{ll} u=\frac{1}{1+y^2} & du=\frac{-2y\,dy}{(1+y^2)^2} \\ dv=dy & v=y \end{array}\right)$$

$$= \frac{y}{1+y^2} + 2\int \frac{y^2\,dy}{(1+y^2)^2} = \frac{y}{1+y^2} + 2\cdot\int \frac{y^2+1-1}{(y^2+1)^2}\,dy$$

$$= \frac{y}{1+y^2} + 2\int \frac{dy}{y^2+1} - 2\int \frac{dy}{(1+y^2)^2}$$

$$2\int \frac{dy}{(1+y^2)^2} = \operatorname{arctg} y + C + \frac{y}{1+y^2}$$

$$\int_{\frac{\sqrt{2}}{2}}^{1} \frac{dy}{(1+y^2)^2} = \frac{1}{2}\left(\operatorname{arctg} y + \frac{y}{1+y^2}\right)\Bigg|_{\frac{\sqrt{2}}{2}}^{1} = \frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}-\operatorname{arctg}\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{1+\frac{1}{2}}\right)$$

$$I = 2\sqrt{2}\cdot\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}-\operatorname{arctg}\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{3}\right)$$

$$= \sqrt{2}\cdot\left(\frac{\pi}{4}+\frac{1}{2}-\operatorname{arctg}\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{3}\right)$$

$$= \frac{\pi\sqrt{2}}{4}+\frac{\sqrt{2}}{2}-\frac{2}{3}-\sqrt{2}\operatorname{arctg}\frac{\sqrt{2}}{2}$$

(4) $$I=\int_0^{\frac{\pi}{2}}\frac{x\,dx}{(\cos x+\sin x)\sqrt{\sin 2x}} = \left(\begin{array}{l} t=\frac{\pi}{2}-x \\ dt=-dx \end{array}\right)$$

$$=\int_{\frac{\pi}{2}}^{0}\frac{\left(\frac{\pi}{2}-t\right)(-dt)}{\left(\cos\left(\frac{\pi}{2}-t\right)+\sin\left(\frac{\pi}{2}-t\right)\right)\sqrt{\sin\left(2\left(\frac{\pi}{2}-t\right)\right)}} = \int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-t\right)dt}{(\sin t+\cos t)\sqrt{\sin(\pi-2t)}}$$

$$=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-t\right)dt}{(\sin t+\cos t)\sqrt{\sin 2t}} = \frac{\pi}{2}\cdot\int_0^{\frac{\pi}{2}}\frac{dt}{(\sin t+\cos t)\sqrt{\sin 2t}} - I$$

$$I=\frac{\pi}{4}\cdot\int_0^{\frac{\pi}{2}}\frac{dt}{(\sin t+\cos t)\sqrt{\sin 2t}} = \left(\begin{array}{lll} y=\operatorname{tg}t & dt=\frac{dy}{1+y^2}, & \cos t=\frac{1}{\sqrt{1+y^2}} \\ t=\operatorname{arctg}y & 1+\operatorname{tg}^2 t=\frac{1}{\cos^2 t}, & \sin t=\frac{y}{\sqrt{1+y^2}} \end{array}\right)$$

$$=\frac{\pi}{4}\cdot\int_0^{+\infty}\frac{\frac{dy}{1+y^2}}{\left(\frac{y}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+y^2}}\right)\sqrt{2\cdot\frac{y}{\sqrt{1+y^2}}\cdot\frac{1}{\sqrt{1+y^2}}}} = \frac{\pi}{4}\int_0^{+\infty}\frac{\frac{dy}{\cancel{1+y^2}}}{\frac{y+1}{\cancel{1+y^2}}\sqrt{2y}}$$

$$=\frac{\pi}{4}\cdot\int_0^{+\infty}\frac{dy}{(y+1)\sqrt{2y}} = \left(\begin{array}{ll} z=\sqrt{2y} & 2dy=2z\,dz \\ z^2=2y & \end{array}\right)$$

$$=\frac{\pi}{4}\cdot\int_0^{+\infty}\frac{\cancel{z}\,dz}{\left(\frac{z^2}{2}+1\right)\cdot\cancel{z}} = \frac{\pi}{4}\int_0^{+\infty}\frac{dz}{1+\left(\frac{z\sqrt{2}}{2}\right)^2} = \frac{\pi}{4}\int_0^{+\infty}\frac{\sqrt{2}\cdot d\left(\frac{z\sqrt{2}}{2}\right)}{1+\left(\frac{z\sqrt{2}}{2}\right)^2}$$

$$=\frac{\pi\sqrt{2}}{4}\operatorname{arctg}\frac{z\sqrt{2}}{2}\Big|_0^{+\infty} = \frac{\pi\sqrt{2}}{4}\cdot(\operatorname{arctg}\infty-0) = \frac{\pi\sqrt{2}}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}\sqrt{2}$$

$$\boxed{I=\frac{\pi^2}{4\sqrt{2}}}$$

⑤ $$I=\int_0^{\pi}\cos x\cdot\ln(5-2\cos^2x+\cos^4x)\,dx$$

$$=\begin{pmatrix} t=\pi-x \\ dt=-dx\end{pmatrix}=\int_{\pi}^{0}\cos(\pi-t)\cdot\ln\left(5-2\cos^2(\pi-t)+\cos^4(\pi-t)\right)(-dt)$$

$$=\int_0^{\pi}(-\cos t)\cdot\ln(5-2\cos^2t+\cos^4t)\,dt=-I$$

$$\Rightarrow\ \boxed{I=0}$$

⑥ Доказати: $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{tgx}}=\int_0^{\frac{\pi}{2}}\sqrt{tgx}\,dx=\frac{\pi}{\sqrt2}$

сингуларитет у 0 сингуларитет у $\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{tgx}}=\begin{pmatrix} y^2=tgx & x=arctg\,y^2 \\ y=\sqrt{tgx} & dx=\frac{1}{1+y^4}\cdot 2y\,dy\end{pmatrix}$$

$$=\int_0^{+\infty}\frac{2y}{1+y^4}\cdot\frac{1}{y}\,dy=2\cdot\int_0^{+\infty}\frac{dy}{1+y^4}$$

$$\int_0^{\frac{\pi}{2}}\sqrt{tgx}\,dx=\begin{pmatrix} y^2=tgx & dx=\frac{1}{1+y^4}\cdot 2y\,dy \\ x=arctg\,y^2 & \end{pmatrix}$$

$$=\int_0^{+\infty}y\cdot\frac{2y}{1+y^4}\,dy=2\cdot\int_0^{+\infty}\frac{y^2}{1+y^4}\,dy=\begin{pmatrix} y=\frac{1}{t} \\ dy=\frac{-1}{t^2}\,dt\end{pmatrix}$$

$$=2\int_{+\infty}^{0}\frac{\frac{1}{t^2}}{1+\frac{1}{t^4}}\cdot\frac{-1}{t^2}\,dt$$

$$=2\cdot\int_0^{+\infty}\frac{dt}{t^4+1}\ \Rightarrow\ \text{једнаки су}$$

$\displaystyle\int_0^{+\infty}\frac{dt}{t^4+1}$ за домаћи (рационална)

II начин (од оног уобичајеног за реш. рацион.)

$$2I=2\int_0^{+\infty}\frac{1+y^2}{1+y^4}\,dy\ \Rightarrow\ I=\int_0^{+\infty}\frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}\,dy$$

смена: $u=y-\frac{1}{y}$, $du=\left(1+\frac{1}{y^2}\right)dy$

$$I=\int_{-\infty}^{+\infty}\frac{du}{u^2+2}=\ldots=\frac{\pi}{\sqrt2}$$