1

$$I=\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x+2}{2-\cos^2 x}\,dx$$

може одмах $t=tg\frac{x}{2}$

али боље је раздвојити на збир 3 инт.

(брже)

$$I=\int_0^{\frac{\pi}{4}}\frac{\sin x\,dx}{2-\cos^2 x}+\int_0^{\frac{\pi}{4}}\frac{\cos x}{2-\cos^2 x}dx+\int_0^{\frac{\pi}{4}}\frac{2}{2-\cos^2 x}dx$$

$t_1=\cos x$, $dt_1=-\sin x\,dx$ $\qquad$ $t_2=\sin x$, $dt_2=\cos x\,dx$ $\qquad$ $t_3=tg x$, $x=arctg\, t_3$, $1+tg^2x=\frac{1}{\cos^2 x}$, $dx=\frac{1}{t_3^2+1}dt_3$, $\cos^2 x=\frac{1}{1+t_3^2}$

$$I=\int_1^{\frac{\sqrt{2}}{2}}\frac{-dt_1}{2-t_1^2}+\int_0^{\frac{\sqrt{2}}{2}}\frac{dt_2}{2-(1-t_2^2)}+2\cdot\int_0^1\frac{dt_3}{(1+t_3^2)\left(2-\frac{1}{1+t_3^2}\right)}$$

$$I=\int_{\frac{\sqrt{2}}{2}}^1\frac{dt_1}{2-t_1^2}+\int_0^{\frac{\sqrt{2}}{2}}\frac{dt_2}{1+t_2^2}+2\int_0^1\frac{dt_3}{2+2t_3^2-1}$$

$$I=\frac{1}{2}\int_{\frac{\sqrt{2}}{2}}^1\frac{\sqrt{2}\,d\left(\frac{t_1}{\sqrt{2}}\right)}{1-\left(\frac{t_1}{\sqrt{2}}\right)^2}+arctg\,t_2\Big|_0^{\frac{\sqrt{2}}{2}}+2\cdot\int_0^1\frac{dt_3}{1+2t_3^2}$$

$$I=\frac{1}{2}\cdot\sqrt{2}\cdot\frac{1}{2}\ln\left|\frac{1+\frac{t_1}{\sqrt{2}}}{1-\frac{t_1}{\sqrt{2}}}\right|\Bigg|_{\frac{\sqrt{2}}{2}}^1+arctg\frac{\sqrt{2}}{2}+2\cdot\int_0^1\frac{\frac{1}{\sqrt{2}}\,d(t_3\sqrt{2})}{1+(t_3\sqrt{2})^2}$$

$$I=\frac{\sqrt{2}}{4}\ln\left|\frac{\sqrt{2}+t_1}{\sqrt{2}-t_1}\right|\Bigg|_{\frac{\sqrt{2}}{2}}^1+arctg\frac{\sqrt{2}}{2}+\sqrt{2}\cdot arctg(t_3\sqrt{2})\Big|_0^1$$

$$I=\frac{\sqrt{2}}{4}\left(\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|-\ln\left|\frac{\sqrt{2}+\frac{\sqrt{2}}{2}}{\sqrt{2}-\frac{\sqrt{2}}{2}}\right|\right)+arctg\frac{\sqrt{2}}{2}+\sqrt{2}\,arctg\sqrt{2}$$

$$I=\frac{\sqrt{2}}{4}\cdot\left(\ln\left|\frac{(\sqrt{2}+1)^2}{(\sqrt{2})^2-1}\right|-\ln\left|\frac{3\sqrt{2}}{\sqrt{2}}\right|\right)+arctg\frac{\sqrt{2}}{2}+\sqrt{2}\,arctg\sqrt{2}$$

$$I=\frac{\sqrt{2}}{4}\left(\ln(3+2\sqrt{2})-\ln 3\right)+arctg\frac{\sqrt{2}}{2}+\sqrt{2}\,arctg\sqrt{2}$$

(2)

$$I=\int_0^{\pi}\ln(\sin x)\,dx=\int_0^{\pi}\ln\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)dx$$

$$=\int_0^{\pi}\left(\ln 2+\ln\left(\sin\frac{x}{2}\right)+\ln\left(\cos\frac{x}{2}\right)\right)dx$$

$$=\ln 2\cdot\pi+\int_0^{\pi}\ln\left(\sin\frac{x}{2}\right)dx+\int_0^{\pi}\ln\left(\cos\frac{x}{2}\right)dx$$

$$t=\frac{x}{2},\quad dt=\frac{dx}{2}$$

$$=\ln 2\cdot\pi+2\int_0^{\frac{\pi}{2}}\ln(\sin t)\,dt+2\int_0^{\frac{\pi}{2}}\ln(\cos t)\,dt$$

$$\int_0^{\frac{\pi}{2}}\ln(\cos t)dt=\left(\begin{array}{l}\text{смена: } u=t+\frac{\pi}{2}\\ du=dt\end{array}\right)$$

$$=\int_{\frac{\pi}{2}}^{\pi}\ln\left(\cos\left(u-\frac{\pi}{2}\right)\right)du=\int_{\frac{\pi}{2}}^{\pi}\ln(\sin u)du=\int_{\frac{\pi}{2}}^{\pi}\ln(\sin t)dt$$

$$I=\ln 2\cdot\pi+2\left(\int_0^{\frac{\pi}{2}}\ln(\sin t)dt+\int_{\frac{\pi}{2}}^{\pi}\ln(\sin t)dt\right)$$

$$I=\ln 2\,\pi+2\cdot\underbrace{\int_0^{\pi}\ln(\sin t)dt}_{I}$$

$$\Rightarrow\ \boxed{I=-\pi\ln 2}$$

додатно.

$$\int_0^{\frac{\pi}{2}}\ln(\sin t)dt=\left(\begin{array}{l}u=t-\frac{\pi}{2}\\ du=dt\end{array}\right)=\int_{-\frac{\pi}{2}}^{0}\ln\left(\sin\left(u+\frac{\pi}{2}\right)\right)du$$

$$=\int_{-\frac{\pi}{2}}^{0}\ln(\cos u)du=\int_0^{\frac{\pi}{2}}\ln(\cos u)du$$

јер је парна ф-ја под инт.

Ако је $I_1=\int_0^{\pi/2}\ln(\sin t)dt$, онда је

$$I=\ln 2\cdot\pi+2I_1+2I_1$$

$$-\pi\ln 2=\ln 2\,\pi+4I_1\quad\Rightarrow\quad I_1=\frac{-2\pi\ln 2}{4}=\frac{-\pi\ln 2}{2}$$

Дакле, $\int_0^{\frac{\pi}{2}}\ln(\sin t)dt=\int_0^{\frac{\pi}{2}}\ln(\cos t)dt=\frac{-\pi\ln 2}{2}$

(3) $$I=\int_0^\pi e^x\cos^2 x\,dx$$

$$\cos^2x+\sin^2x=1$$

$$\cos^2x-\sin^2x=\cos 2x$$

$$\int_0^\pi e^x\cos^2x\,dx+\underbrace{\int_0^\pi e^x\sin^2x\,dx}_{=J}=\int_0^\pi e^x\,dx=e^x\Big|_0^\pi=e^\pi-1=I+J$$

$$I-J=\int_0^\pi e^x(\cos^2x-\sin^2x)\,dx=\int_0^\pi e^x\cos 2x\,dx$$

$$=\begin{pmatrix} u=e^x & du=e^x dx\\ dv=\cos 2x\,dx & v=\sin 2x\cdot\frac12\end{pmatrix}=\underbrace{e^x\,\frac12\sin 2x\Big|_0^\pi}_{=0}-\int_0^\pi\frac12\sin 2x\,e^x\,dx$$

$$\begin{array}{l} u=e^x\\ dv=\sin 2x\,dx\\ du=e^x dx\\ v=-\cos 2x\,\frac12\end{array}$$

$$=-\frac12\left(-e^x\cos 2x\cdot\frac12\Big|_0^\pi+\frac12\int_0^\pi\cos 2x\,e^x\,dx\right)$$

$$=\frac14e^\pi-\frac14-\frac14\cdot\int_0^\pi\cos 2x\,e^x\,dx \qquad /\cdot 4$$

$$5\cdot\int_0^\pi\cos 2x\,e^x\,dx=e^\pi-1$$

$$I-J=\frac{e^\pi-1}{5}$$

$$\Rightarrow\quad 2I=\frac{e^\pi-1}{5}+e^\pi-1=\frac{6(e^\pi-1)}{5}$$

$$\boxed{I=\frac35(e^\pi-1)}$$

II начин: $\cos 2x=\mathrm{Re}\,e^{2ix}$

$$\int_0^\pi e^x\cos 2x\,dx=\int_0^\pi e^x\,\mathrm{Re}\,e^{2ix}\,dx=\mathrm{Re}\int_0^\pi e^{x(1+2i)}\,dx$$

посматрамо овај компл. број као конст!

$$=\mathrm{Re}\left(\frac{e^{x(1+2i)}}{1+2i}\Big|_0^\pi\right)=\mathrm{Re}\left(\frac{e^{\pi(1+2i)}-e^0}{1+2i}\right)$$

$$=\mathrm{Re}\,\frac{e^\pi\cdot e^{2i\pi}-1}{1+2i}=\mathrm{Re}\,\frac{e^\pi-1}{1+2i}\,\frac{1-2i}{1-2i}$$

$$=\mathrm{Re}\,\frac{e^\pi-1-2i(e^\pi-1)}{5}=\frac{e^\pi-1}{5}$$

(4)

$$I=\int_0^{\pi}\cos^{n-1}x\cdot\sin(n+1)x\,dx$$

$$\sin(n+1)x=\sin nx\cdot\cos x+\sin x\cdot\cos nx$$

$$I=\underbrace{\int_0^{\pi}\cos^{n-1}x\cdot\sin nx\cdot\cos x\,dx}+\int_0^{\pi}\cos^{n-1}x\cdot\sin x\cdot\cos nx\,dx$$

$$\int_0^{\pi}\cos^{n-1}x\cdot\cos nx\cdot\sin x\,dx=\left(\begin{array}{ll}u=\cos nx & du=-\sin nx\,n\,dx\\ dv=\cos^{n-1}x\cdot\sin x\,dx & \\ & v=-\dfrac{\cos^n x}{n}\end{array}\right.$$

$$=\cos nx\cdot\frac{-\cos^n x}{n}\Big|_0^{\pi}-\int_0^{\pi}\cos^n x\cdot\sin nx\,dx$$

$$=\cos n\pi\cdot\frac{-1}{n}\cos^n\pi+\frac{1}{n}-\int_0^{\pi}\cos^n x\cdot\sin nx\,dx$$

$$=(-1)^n\cdot\frac{-1}{n}\cdot(-1)^n+\frac{1}{n}-\int_0^{\pi}\cos^n x\cdot\sin nx\,dx$$

$$=\cancel{\frac{-1}{n}}+\cancel{\frac{1}{n}}-\underbrace{\int_0^{\pi}\cos^n x\cdot\sin nx\,dx}$$

$$\Rightarrow\boxed{I=0}$$

Напомена: може парц. одмах
$u=\sin^{n-1}x$
$dv=\cos(n+1)x\,dx$
па после опет парц. овако!

(5)

$$I=\int_0^{\pi}\sin^{n-1}x\,\cos(n+1)x\,dx$$

$$I=\int_0^{\pi}\sin^{n-1}x\,(\cos nx\cdot\cos x-\sin nx\cdot\sin x)\,dx$$

$$=\underbrace{\int_0^{\pi}\sin^{n-1}x\cdot\cos nx\cdot\cos x\,dx}_{J}-\int_0^{\pi}\sin^n x\cdot\sin nx\,dx$$

$$u=\cos nx\qquad du=-\sin nx\cdot n\,dx$$
$$dv=\sin^{n-1}x\cos x\,dx\qquad v=\frac{\sin^n x}{n}$$

$$J=\underbrace{\cos nx\cdot\frac{\sin^n x}{n}\Big|_0^{\pi}}_{=0}+\int_0^{\pi}\sin^n x\cdot\sin nx\,dx=\int_0^{\pi}\sin^n x\cdot\sin nx\,dx$$

$$\Rightarrow\boxed{I=0}$$