① $$I=\int_{-1}^{1}\frac{dx}{(1+x^2)(1+e^x)}=\int_{-1}^{0}\frac{dx}{(1+x^2)(1+e^x)}+\int_{0}^{1}\frac{dx}{(1+x^2)(1+e^x)}$$

смена: $t=-x$, $dt=-dx$

$$=\int_{1}^{0}\frac{-dt}{(1+t^2)(1+e^{-t})}+\int_{0}^{1}\frac{dx}{(1+x^2)(1+e^x)}=\int_{0}^{1}\frac{dx}{(1+x^2)(1+e^{-x})}+\int_{0}^{1}\frac{dx}{(1+x^2)(1+e^x)}$$

$$=\int_{0}^{1}\frac{e^x\,dx}{(1+x^2)(e^x+1)}+\int_{0}^{1}\frac{dx}{(1+x^2)(1+e^x)}=\int_{0}^{1}\frac{dx}{1+x^2}$$

$$=\operatorname{arctg}x\Big|_0^1=\operatorname{arctg}1-\operatorname{arctg}0=\frac{\pi}{4}$$

② Одредити $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n-1}\frac{n}{n^2+k^2}$ користећи дефиницију одређеног интеграла.

$$\sum_{k=1}^{n-1}\frac{n}{n^2+k^2}=\sum_{k=1}^{n-1}\frac{1}{1+\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}$$

овде препознајемо интегралну суму за ф-ју $\frac{1}{1+x^2}$ и поделу $P=\{0,\frac{1}{n},\frac{2}{n},\dots,\frac{n-1}{n},1\}$ сегмента $[0,1]$ са истакнутим тачкама $\{\frac{1}{n},\frac{2}{n},\dots,\frac{n-1}{n},1\}$ и $\lambda(P)=\frac{1}{n}$

$$f(x)=\frac{1}{1+x^2}$$

$$\int_0^1\frac{dx}{1+x^2}=\lim_{n\to\infty}\sum_{k=1}^{n}f(\xi_k)(x_k-x_{k-1})$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}$$

$$=\lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\frac{1}{1+\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}+\frac{1}{2}\cdot\frac{1}{n}\right)$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{1}{1+\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}$$

Пошто знамо: $\int_0^1\frac{dx}{1+x^2}=\operatorname{arctg}x\Big|_0^1=\frac{\pi}{4}$ задатак је завршен.

(3) $I = \int_0^{\pi} \frac{1+\sin^2 x}{6-\cos^2 x+|\cos x|}\cdot \sin x \cdot x\,dx = \left(\begin{array}{l} t = \pi - x \\ dt = -dx \end{array} \right)$

$$I = \int_{\pi}^{0} \frac{1+\sin^2 t}{6-\cos^2 t+|\cos t|} \sin t\,(\pi - t)(-dt)$$

$$= \int_0^{\pi} \frac{1+\sin^2 t}{6-\cos^2 t+|\cos t|} \sin t \cdot \pi\,dt - I$$

↑ узимамо јер је
$\sin(\pi - x) = \sin x$
$\cos^2(\pi - x) = \cos^2 x$
$|\cos(\pi - x)| = |\cos x|$
па је израз без x
исти после смене!

$$2I = \pi \cdot \int_0^{\pi} \frac{(1+\sin^2 t)\cdot \sin t}{6-\cos^2 t+|\cos t|}\,dt$$

↑ овде може смена $\cos t = u$
$du = -\sin t\,dt$

$$I = \frac{\pi}{2}\cdot \int_1^{-1} \frac{(1+1-u^2)\cdot(-du)}{6-u^2+|u|} = \frac{\pi}{2}\cdot \int_{-1}^{1} \frac{(2-u^2)\,du}{6-u^2+|u|}$$

парна ф-ја
$[-1,1]$ симетричан интервал

$$= \frac{\pi}{2}\cdot 2\cdot \underbrace{\int_0^1 \frac{2-u^2}{6-u^2+u}\,du}$$

интеграл рационалне ф-је (доначи)

$a \neq 1$
$a > 0$

(4) $I = \int_{\frac{1}{a}}^{a} \frac{|\ln x|}{1+x}\,dx = \left(\begin{array}{l} \text{смена } t = \frac{1}{x},\ x = \frac{1}{t} \\ dx = \frac{-1}{t^2}\,dt \end{array} \right)$

$$= \int_a^{\frac{1}{a}} \frac{|\ln \frac{1}{t}|}{1+\frac{1}{t}}\,\frac{-1}{t^2}\,dt = \int_{\frac{1}{a}}^{a} \frac{|-\ln t|}{1+\frac{1}{t}}\,\frac{dt}{t^2}$$

$$= \int_{\frac{1}{a}}^{a} \frac{|\ln t|}{t(t+1)}\,dt = \int_{\frac{1}{a}}^{a} \left(\frac{|\ln t|}{t} - \frac{|\ln t|}{t+1} \right) dt$$

$$= \int_{\frac{1}{a}}^{a} \frac{|\ln t|}{t}\,dt - \underbrace{\int_{\frac{1}{a}}^{a} \frac{|\ln t|}{t+1}\,dt}_{I}$$

↑ можемо да раздвојимо јер оба инт постоје

$$2I = \int_{\frac{1}{a}}^{a} \frac{|\ln t|}{t}\,dt$$

за $a>1$: $$2I = \int_{\frac{1}{a}}^{1} \frac{|\ln t|}{t}\,dt + \int_{1}^{a} \frac{|\ln t|}{t}\,dt$$

$$= \int_{\frac{1}{a}}^{1} \frac{-\ln t}{t}\,dt + \int_{1}^{a} \frac{\ln t}{t}\,dt$$

$$= -\frac{\ln^2 t}{2}\Big|_{\frac{1}{a}}^{1} + \frac{\ln^2 t}{2}\Big|_{1}^{a} = -\frac{1}{2}\left(\overset{0}{\ln^2 1} - \ln^2\frac{1}{a}\right) + \frac{1}{2}\left(\ln^2 a - 0\right)$$

$$\int \frac{\ln t}{t}\,dt = \left(\begin{matrix} u=\ln t \\ du=\frac{1}{t}dt \end{matrix}\right) = \int u\,du = \frac{u^2}{2}+C = \frac{\ln^2 t}{2}+C$$

$$2I = \frac{1}{2}\ln^2\frac{1}{a} + \frac{1}{2}\ln^2 a = \ln^2 a \qquad \left(\ln\frac{1}{a} = -\ln a\right)$$

$$\boxed{I = \frac{1}{2}\ln^2 a}$$

за $0<a<1$: $$2I = -\int_{a}^{\frac{1}{a}} \frac{|\ln t|}{t}\,dt = -\left(\int_{a}^{1} \frac{|\ln t|}{t}\,dt + \int_{1}^{\frac{1}{a}} \frac{|\ln t|}{t}\,dt\right)$$

$$= -\left(\int_{a}^{1} \frac{-\ln t}{t}\,dt + \int_{1}^{\frac{1}{a}} \frac{\ln t}{t}\,dt\right)$$

$$= \int_{a}^{1} \frac{\ln t}{t}\,dt - \int_{1}^{\frac{1}{a}} \frac{\ln t}{t}\,dt$$

$$= \frac{\ln^2 t}{2}\Big|_{a}^{1} - \frac{\ln^2 t}{2}\Big|_{1}^{\frac{1}{a}}$$

$$= -\frac{\ln^2 a}{2} - \ln^2\frac{1}{a}\cdot\frac{1}{2} = -\ln^2 a$$

$$\boxed{I = -\frac{1}{2}\ln^2 a}$$

(5) Доказати једнакост $\int_0^1 \frac{\cos^2\frac{\pi x}{2}}{\sqrt{x-x^2}}dx = \int_0^1 \frac{\sin^2\frac{\pi x}{2}}{\sqrt{x-x^2}}dx$, а затим израчунати интеграле.

$$\int_0^1 \frac{\cos^2\frac{\pi x}{2}}{\sqrt{x-x^2}}dx = \left(\begin{array}{ll} \text{смена: } & t = 1-x \\ & dt = -dx \end{array} \right)$$

$$= \int_1^0 \frac{\cos^2\left(\frac{\pi}{2}(1-t)\right)}{\sqrt{1-t-(1-t)^2}}(-dt) = \int_0^1 \frac{\cos^2\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)}{\sqrt{-t+2t-t^2}}dt$$

$$= \int_0^1 \frac{\left(\sin\frac{\pi t}{2}\right)^2}{\sqrt{t-t^2}}dt$$

$$I = \int_0^1 \frac{\cos^2\frac{\pi x}{2}}{\sqrt{x-x^2}}dx = \int_0^1 \frac{\sin^2\frac{\pi x}{2}}{\sqrt{x-x^2}}dx$$

$$\Rightarrow 2I = \int_0^1 \frac{\cos^2\frac{\pi x}{2}+\sin^2\frac{\pi x}{2}}{\sqrt{x-x^2}}dx = \int_0^1 \frac{dx}{\sqrt{x-x^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \left(\begin{array}{ll} t=\sqrt{x} & 2t\,dt = dx \\ t^2 = x & \end{array} \right)$$

$$= \int_0^1 \frac{2\cancel{t}\,dt}{\cancel{t}\cdot\sqrt{1-t^2}} = 2\arcsin t\,\Big|_0^1 = 2\cdot\frac{\pi}{2} = \pi$$

$$\Rightarrow I = \frac{\pi}{2}$$

(6) $I_1 = \int_0^{\pi} (x \sin x)^2 dx$

$I_2 = \int_0^{\pi} (x \cdot \cos x)^2 dx$

$I_1 + I_2 = \int_0^{\pi} (x^2 \sin^2 x + x^2 \cos^2 x) dx = \int_0^{\pi} x^2 dx = \frac{x^3}{3} \Big|_0^{\pi} = \frac{\pi^3}{3}$

$I_2 - I_1 = \int_0^{\pi} x^2 (\cos^2 x - \sin^2 x) dx = \int_0^{\pi} x^2 \cdot \cos 2x \, dx = \left(\begin{array}{ll} u = x^2 & du = 2x\,dx \\ dv = \cos 2x\,dx & v = \frac{\sin 2x}{2} \end{array} \right)$

$= x^2 \cdot \frac{\sin 2x}{2} \Big|_0^{\pi} - \int_0^{\pi} x \sin 2x \, dx = \left(\begin{array}{ll} u = x & du = dx \\ dv = \sin 2x\,dx & v = \frac{-\cos 2x}{2} \end{array} \right)$

$= \frac{\pi^2}{2} \cdot \sin 2\pi - 0 - \left(-\frac{\cos 2x}{2} \cdot x \Big|_0^{\pi} + \int_0^{\pi} \frac{1}{2} \cos 2x \, dx \right)$

$= \frac{\cos 2\pi}{2} \cdot \pi - 0 - \frac{1}{2} \cdot \frac{\sin 2x}{2} \Big|_0^{\pi}$

$= \frac{\pi}{2} - \frac{1}{4} \cdot (\underbrace{\sin 2\pi}_{0} - \underbrace{\sin 0}_{0}) = \frac{\pi}{2}$

$\Rightarrow \quad 2I_2 = \frac{\pi^3}{3} + \frac{\pi}{2} \qquad 2I_1 = \frac{\pi^3}{3} - \frac{\pi}{2}$

$I_2 = \frac{1}{2}\pi \left(\frac{\pi^2}{3} + \frac{1}{2} \right), \quad I_1 = \frac{\pi}{2} \left(\frac{\pi^2}{3} - \frac{1}{2} \right)$