$$I=\frac{1}{2}(1-x)^2\arcsin(1-x)-\frac{1}{4}\arcsin(1-x)+\frac{1}{4}(1-x)\sqrt{1-(1-2x+x^2)}+C$$ (са прошлог часа резул.)

(1)

$$\int\ln(\sqrt{1-x}+\sqrt{1+x})\,dx=\begin{pmatrix} u=\ln(\sqrt{1-x}+\sqrt{1+x}) & du=\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\cdot\left(\frac{1}{2}\frac{-1}{\sqrt{1-x}}+\frac{1}{2}\frac{1}{\sqrt{1+x}}\right)dx \\ dv=dx & v=x\end{pmatrix}$$

$$=x\ln(\sqrt{1-x}+\sqrt{1+x})+\int\frac{x}{\sqrt{1-x}+\sqrt{1+x}}\frac{1}{2}\cdot\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1-x}\sqrt{1+x}}dx$$

$$=x\ln(\sqrt{1-x}+\sqrt{1+x})+\frac{1}{2}\int\frac{x\,(\sqrt{1+x}-\sqrt{1-x})}{\sqrt{1-x^2}\,(\sqrt{1-x}+\sqrt{1+x})}\cdot\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}dx$$

$$=x\ln(\sqrt{1-x}+\sqrt{1+x})+\frac{1}{2}\int\frac{x}{\sqrt{1-x^2}}\cdot\frac{(1+\cancel{x}-2\sqrt{1-x^2}+1-\cancel{x})}{(1+x-(1-x))}dx$$

$$=x\ln(\sqrt{1-x}+\sqrt{1+x})+\frac{1}{2}\int\frac{\cancel{x}}{\sqrt{1-x^2}}\cdot\frac{\cancel{2}(1-\sqrt{1-x^2})}{\cancel{2x}}dx$$

$$=x\ln(\sqrt{1-x}+\sqrt{1+x})+\frac{1}{2}\left(\int\frac{dx}{\sqrt{1-x^2}}-\int dx\right)$$

$$=x\ln(\sqrt{1-x}+\sqrt{1+x})+\frac{1}{2}\arcsin x-\frac{1}{2}\cdot x+C$$

(2)

$$\int\frac{dx}{(\sqrt{x+2}+1)\sqrt{\sqrt{x+2}-1}}=\begin{pmatrix} t=\sqrt{x+2} & dx=2t\,dt \\ t^2=x+2 & \end{pmatrix}$$

$$=\int\frac{2t\,dt}{(t+1)\sqrt{t-1}}=\begin{pmatrix} u^2=t-1 \\ 2u\,du=dt\end{pmatrix}=\int\frac{2(u^2+1)\,2\cancel{u}\,du}{(u^2+2)\cdot\cancel{u}}$$

$$=4\int\frac{(u^2+1)\,du}{u^2+2}=4\int\frac{u^2+2-1}{u^2+2}du=4\left(u-\int\frac{du}{u^2+2}\right)$$

$$=4\left(u-\frac{1}{2}\int\frac{du}{(\frac{u}{\sqrt{2}})^2+1}\right)=4\left(u-\frac{1}{2}\int\frac{d(\frac{u}{\sqrt{2}})\cdot\sqrt{2}}{(\frac{u}{\sqrt{2}})^2+1}\right)$$

$$=4u-4\frac{\sqrt{2}}{2}\cdot\operatorname{arctg}\frac{u}{\sqrt{2}}+C=4\sqrt{\sqrt{x+2}-1}-2\sqrt{2}\cdot\operatorname{arctg}\frac{\sqrt{\sqrt{x+2}-1}}{\sqrt{2}}+C$$

(3)

$$\int\frac{\sqrt{x+1}}{(\sqrt{x+1}-1)^2}dx=\begin{pmatrix} t=\sqrt{x+1} & 2t\,dt=dx \\ t^2=x+1 & \end{pmatrix}=\int\frac{t\cdot 2t\,dt}{(t-1)^2}$$

$$=\int\frac{2t^2dt}{(t-1)^2}=\int\frac{2(t^2-2t+1)+4t-2}{t^2-2t+1}dt=\int 2\,dt+2\int\frac{2t-1}{t^2-2t+1}dt$$

$$\begin{matrix} m=t^2-2t+1 \\ dm=(2t-2)\,dt\end{matrix}$$

$$=2\cdot t+2\cdot\int\frac{2t-2+1}{t^2-2t+1}dt=2t+2\left(\int\frac{dm}{m}+\int\frac{dt}{(t-1)^2}\right)$$

$$=2t+2\ln|m|+\frac{-1}{t-1}+C=2t+2\ln\overset{2\ln|t-1|}{\overset{\|}{|t^2-2t+1|}}-\frac{2}{t-1}+C=2\cdot\sqrt{x+1}+4\ln|\sqrt{x+1}-1|-\frac{2}{\sqrt{x+1}-1}+C$$

④ $a>0$

$$I=\int \frac{x\,dx}{\sqrt[4]{x^3(a-x)}} = \int \frac{x\,dx}{\sqrt[4]{x^4\cdot\frac{a-x}{x}}} = \int \frac{\cancel{x}^{\,1}\,dx}{\cancel{|x|}_{\,1}\,\sqrt[4]{\frac{a-x}{x}}}$$

x	$-$	$+$	$+$
$a-x$	$+$	$+$	$-$
$\frac{a-x}{x}$	$-$	$+$	$-$

The column boundaries are 0 and a.

За $x\in(0,a)$ је деф. $|x|=x$ тада

$$I=\int \sqrt[4]{\frac{x}{a-x}}\,dx$$

смена $t=\sqrt[4]{\frac{x}{a-x}}$

$t^4(a-x)=x$

$t^4 a = x(1+t^4)$ $\quad x=\frac{at^4}{1+t^4}=\frac{a(t^4+1)-a}{1+t^4}$

$$dx=-a\cdot\frac{-4t^3\,dt}{(1+t^4)^2}=\frac{4at^3\,dt}{(1+t^4)^2}$$

$$I=\int t\cdot\frac{4at^3\,dt}{(1+t^4)^2}=4a\int\frac{t^4\,dt}{(1+t^4)^2}=4a\int\frac{t^4+1-1}{(t^4+1)^2}\,dt$$

$$I=4a\left(\int\frac{dt}{t^4+1}-\int\frac{dt}{(t^4+1)^2}\right)$$

$J=\int\frac{dt}{t^4+1}$ (овај смо рачунали на вежбама)

$\int\frac{dt}{(t^4+1)^2}=?$ добићемо га преко J !

$$J=\int\frac{dt}{t^4+1}=\left(\begin{array}{ll} u=\frac{1}{t^4+1} & du=\frac{-4t^3\,dt}{(t^4+1)^2} \\ dv=dt & v=t\end{array}\right)$$

$$J=\frac{t}{t^4+1}+\underbrace{4\int\frac{t^4\,dt}{(t^4+1)^2}}_{=\ \frac{1}{a}\cdot I}$$

$$\frac{1}{a}I=J-\frac{t}{t^4+1}$$

$$\boxed{I=aJ-\frac{at}{t^4+1}}$$

и потребно је још на крају све изразити преко x (заменом $t=\sqrt[4]{\frac{x}{a-x}}$)

⑤ $$I=\int\frac{dx}{(x-a)^{\frac{n+1}{n}}\cdot(x-b)^{\frac{n-1}{n}}}\quad, a\neq b$$

$$I=\int\frac{dx}{(x-a)(x-b)\cdot\frac{(x-a)^{1/n}}{(x-b)^{1/n}}}=\int\frac{dx}{(x-a)(x-b)\sqrt[n]{\frac{x-a}{x-b}}}$$

смена: $t=\sqrt[n]{\frac{x-a}{x-b}}$

$$t^n=\frac{x-a}{x-b}$$

$$x-a=t^n(x-b)$$

$$x(1-t^n)=a-bt^n$$

$$x=\frac{a-bt^n}{1-t^n}$$

$$dx=\frac{-bnt^{n-1}(1-t^n)+nt^{n-1}(a-bt^n)}{(1-t^n)^2}dt$$

$$dx=\frac{-bnt^{n-1}+\cancel{bnt^{2n-1}}+nat^{n-1}-\cancel{nbt^{2n-1}}}{(1-t^n)^2}dt$$

$$dx=\frac{n(a-b)t^{n-1}}{(t^n-1)^2}dt$$

$$x-b=\frac{a-bt^n}{1-t^n}-b=\frac{a-\cancel{bt^n}-b+\cancel{bt^n}}{1-t^n}=\frac{a-b}{1-t^n}$$

$$x-a=t^n(x-b)=t^n\cdot\frac{a-b}{1-t^n}$$

$$I=\int\frac{\frac{n\cancel{(a-b)}t^{n-1}}{\cancel{(t^n-1)^2}}dt}{t^n\frac{a-b}{\cancel{1-t^n}}\cdot\frac{\cancel{a-b}}{\cancel{1-t^n}}\cdot t}=\int\frac{n\cancel{t^{n-1}}dt}{(a-b)\,t^{\cancel{n+1}\,2}}$$

$$I=\frac{n}{a-b}\int\frac{dt}{t^2}=\frac{n}{a-b}\cdot\frac{-1}{t}+C$$

$$\boxed{I=\frac{n}{a-b}\cdot\frac{-1}{\sqrt[n]{\frac{x-a}{x-b}}}+C}$$

⑥ $$I=\int\frac{\sin x\,dx}{\sqrt{2+\sin 2x}}=\int\frac{\sin x\,dx}{\sqrt{2+2\sin x\cos x}}$$

1) $2+2\sin x\cos x=2+(\sin x+\cos x)^2-1=1+(\sin x+\cos x)^2$, $t_1=\sin x+\cos x$, $dt_1=(\cos x-\sin x)dx$

2) $2+2\sin x\cos x=2-(\sin x-\cos x)^2+1=3-(\sin x-\cos x)^2$, $t_2=\sin x-\cos x$, $dt_2=(\cos x+\sin x)dx$

$$I=\int\frac{\sin x+\cos x+\sin x-\cos x}{2\sqrt{2+2\sin x\cos x}}dx=\int\frac{\sin x+\cos x}{2\sqrt{3-(\sin x-\cos x)^2}}dx+\int\frac{\sin x-\cos x}{2\sqrt{1+(\sin x-\cos x)^2}}dx$$

$$I=\frac{1}{2}\left(\int\frac{dt_2}{\sqrt{3-t_2^2}}+\int\frac{-dt_1}{\sqrt{1+t_1^2}}\right)=\frac{1}{2}\left(\int\frac{dt_2}{\sqrt3\sqrt{1-\left(\frac{t_2}{\sqrt3}\right)^2}}-\int\frac{dt_1}{\sqrt{1+t_1^2}}\right)$$

$$I=\frac{1}{2}\left(\arcsin\frac{t_2}{\sqrt3}-\ln\left(t_1+\sqrt{t_1^2+1}\right)\right)+C$$

$$\boxed{I=\frac{1}{2}\left(\arcsin\frac{\sin x-\cos x}{\sqrt3}-\ln\left(\sin x+\cos x+\sqrt{2+2\sin x\cos x}\right)\right)+C}$$