Практикум 1

пт. $a>0$

① $I=\int\sqrt{a^2+x^2}\,dx = ?$

I начин: (парцијална инт.) $I=\int\sqrt{a^2+x^2}\,dx = \left(\begin{matrix} u=\sqrt{a^2+x^2} & dv=dx \\ du=\frac{1}{\cancel{2}}\cdot\frac{1}{\sqrt{a^2+x^2}}\cdot\cancel{2}x\,dx & v=x\end{matrix}\right) = x\cdot\sqrt{a^2+x^2}-\int\frac{x^2}{\sqrt{a^2+x^2}}\,dx$

$$=x\sqrt{a^2+x^2}-\int\frac{x^2+a^2-a^2}{\sqrt{a^2+x^2}}\,dx = x\sqrt{a^2+x^2}-\Big(\underbrace{\int\sqrt{a^2+x^2}\,dx}_{I}-\int\frac{a^2dx}{\sqrt{a^2(1+\frac{x}{a})^2}}\Big)$$

$$=x\sqrt{a^2+x^2}-I+\int\frac{a^2\,dx}{|a|\sqrt{1+(\frac{x}{a})^2}}$$

смена: $t=\frac{x}{a}$

$dt=\frac{1}{a}dx$

$$2I=x\sqrt{a^2+x^2}+\int\frac{a^2\cdot\cancel{a}\,dt}{\cancel{|a|}\cdot\sqrt{1+t^2}} = x\sqrt{a^2+x^2}+a^2\cdot\ln|t+\sqrt{1+t^2}|+C_1$$

$a>0$

$$I=\frac{1}{2}\left(x\sqrt{a^2+x^2}+a^2\cdot\ln\left|\frac{x}{a}+\sqrt{1+\frac{x^2}{a^2}}\right|\right)+C$$

II начин: смена $x=a\cdot tg\,t$ jер je $1+tg^2t=\frac{1}{\cos^2t}$

$$I=\int\sqrt{a^2+x^2}\,dx=\left(\begin{matrix} x=a\,tg\,t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2}) \\ dx=a\cdot\frac{1}{\cos^2t}\,dt\end{matrix}\right)=\int\sqrt{a^2+a^2tg^2t}\,\frac{a\,dt}{\cos^2t}$$

$$=\int a\sqrt{1+tg^2t}\cdot\frac{a\,dt}{\cos^2t}=\int a^2\frac{1}{|\cos t|}\frac{dt}{\cos^2t}=a^2\cdot\int\frac{dt}{\cos^3t}=a^2\int\frac{\cos t\,dt}{(1-\sin^2t)^2}$$

$t\in(-\frac{\pi}{2},\frac{\pi}{2})$, $\cos t>0$

смена: $u=\sin t$, $du=\cos t\,dt$

$$=a^2\cdot\int\frac{du}{(1-u^2)^2}$$

знамо: $\int\frac{dt}{1-t^2}=\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|+C=J$

$$J=\int\frac{dt}{1-t^2}=\left(\begin{matrix} u=\frac{1}{1-t^2} & du=\frac{2t}{(1-t^2)^2}dt \\ dv=dt & v=t\end{matrix}\right)=t\cdot\frac{1}{1-t^2}-\int\frac{2t^2}{(1-t^2)^2}dt$$

$$=\frac{t}{1-t^2}-2\cdot\int\frac{t^2-1+1}{(1-t^2)^2}dt=\frac{t}{1-t^2}-2\cdot\Big(\underbrace{\int\frac{-dt}{1-t^2}}_{-J}+\int\frac{dt}{(1-t^2)^2}\Big)$$

$$2\int\frac{dt}{(1-t^2)^2}=J+\frac{t}{1-t^2}=\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|+\frac{t}{1-t^2}+C_1$$

$$\int\frac{dt}{(1-t^2)^2}=\frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|+\frac{t}{2(1-t^2)}+C$$

$$I = a^2 \cdot \left(\frac{1}{4} \ln \left| \frac{1+u}{1-u} \right| + \frac{u}{2(1-u^2)} \right) + C_1$$

$u = \sin t$, $x = a\,\mathrm{tg}\,t$ $\quad \mathrm{tg}\,t = \frac{\sin t}{\cos t}$ $\quad \cos t = \frac{1}{\sqrt{1+\mathrm{tg}^2 t}} = \frac{1}{\sqrt{1+\frac{x^2}{a^2}}}$

$$\sin t = \mathrm{tg}\,t \cdot \cos t = \frac{x}{a} \cdot \frac{|a|}{\sqrt{a^2+x^2}} \underset{a>0}{\overset{\uparrow}{=}} \frac{x}{\sqrt{a^2+x^2}}$$

$$\Rightarrow u = \frac{x}{\sqrt{a^2+x^2}}$$

$$I = a^2 \left(\frac{1}{4} \ln \left| \frac{1+\frac{x}{\sqrt{a^2+x^2}}}{1-\frac{x}{\sqrt{a^2+x^2}}} \right| + \frac{\frac{x}{\sqrt{a^2+x^2}}}{2\cdot\left(1-\frac{x^2}{a^2+x^2}\right)} \right) + C_1 = a^2 \left(\frac{1}{4} \ln \left| \frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}-x} \right| + \frac{x\sqrt{a^2+x^2}}{2(a^2+\cancel{x^2}-\cancel{x^2})} \right) + C_1$$

$$I = a^2 \left(\frac{1}{4} \ln \left| \frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}-x} \cdot \frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}+x} \right| + \frac{x\sqrt{a^2+x^2}}{2a^2} \right) + C_1$$

$$I = \frac{a^2}{4} \ln \left| \frac{(x+\sqrt{a^2+x^2})^2}{a^2+\cancel{x^2}-\cancel{x^2}} \right| + \frac{1}{2} x\sqrt{a^2+x^2} + C_1$$

$$I = \frac{a^2}{4} \cdot \ln \frac{(x+\sqrt{a^2+x^2})^2}{|a^2|} + \frac{x}{2}\sqrt{a^2+x^2} + C_1$$

$$\boxed{I = \frac{a^2}{2} \ln \frac{|x+\sqrt{a^2+x^2}|}{a} + \frac{x}{2}\sqrt{a^2+x^2} + C_1}$$

$$\ln \left| \frac{x}{a} + \sqrt{1+\left(\frac{x}{a}\right)^2} \right| = \ln \frac{|x+\sqrt{a^2+x^2}|}{a}$$

(па је добијен исти резултат наравно)

III начин:

Због $ch^2 t - sh^2 t = 1$ је $ch^2 t = 1 + sh^2 t$, па је згодно узети смену $x = a\,sh\,t$

$$I = \int \sqrt{a^2+x^2}\,dx \underset{\substack{x = a\,sh\,t \\ dx = a\,ch\,t}}{\overset{\uparrow}{=}} \int \sqrt{a^2+a^2 sh^2 t} \cdot a\,ch\,t\,dt = \int a \cdot ch\,t \cdot a\,ch\,t\,dt = \int a^2\,ch^2 t\,dt$$

$$ch\,2t = ch^2 t + sh^2 t = ch^2 t + ch^2 t - 1$$

$$ch^2 t = \frac{1+ch\,2t}{2}$$

$$= a^2 \int \frac{1+ch\,2t}{2}\,dt = a^2\left(\frac{1}{2} t + \frac{1}{4}\,sh\,2t \right) + C$$

$$,\ sh\,2t = 2\,sh\,t\,ch\,t$$

$sh\,t = \frac{x}{a}$, $ch\,t = \sqrt{1+\left(\frac{x}{a}\right)^2}$, $sh\,2t = 2 \cdot \frac{x}{a} \cdot \sqrt{1+\left(\frac{x}{a}\right)^2} = \frac{2x}{a^2}\sqrt{a^2+x^2}$

$$\frac{x}{a} = sh\,t = \frac{e^t - e^{-t}}{2} = u \qquad e^t - e^{-t} = 2 \cdot u \quad / e^t$$

$$(e^t)^2 - 2ue^t - 1 = 0 \qquad e^t_{1,2} = \frac{2u \pm \sqrt{4u^2+4}}{2} = u \pm \sqrt{u^2+1}$$

$$\boxed{t = \mathrm{arsh}\,\frac{x}{a}}$$

$$e^t > 0\,! \Rightarrow e^t = u + \sqrt{u^2+1}$$

$\Rightarrow t = \ln\left(u+\sqrt{u^2+1}\right)$, $\operatorname{arcsh} u = \ln\left(u+\sqrt{u^2+1}\right)$

Дакле, $t = \operatorname{arcsh}\frac{x}{a} = \ln\left(\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2+1}\right) = \ln\frac{x+\sqrt{a^2+x^2}}{a}$ ← овде ч не мора апс. јер је увек >0!? (ни раније)

$$\Rightarrow \quad I = a^2\cdot\left(\frac{1}{2}\ln\frac{x+\sqrt{a^2+x^2}}{a} + \frac{1}{2}\frac{x}{a^2}\sqrt{a^2+x^2}\right)+C$$

$$\boxed{I = \frac{a^2}{2}\ln\frac{x+\sqrt{a^2+x^2}}{a} + \frac{1}{2}x\sqrt{a^2+x^2}+C}$$

② $I = \int\sqrt{a^2-x^2}\,dx = ?$ ($a>0$ по пп.)

<u>I начин</u>: парцијалном интеграцијом

$$I = \int\sqrt{a^2-x^2}\,dx = \left(\begin{array}{ll} u=\sqrt{a^2-x^2} & du = \frac{1}{2}\cdot\frac{1}{\sqrt{a^2-x^2}}\cdot(-2x)\,dx = \frac{-x}{\sqrt{a^2-x^2}}\,dx \\ dv = dx & v = x \end{array}\right)$$

$$= x\sqrt{a^2-x^2} - \int -\frac{x^2}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} + \int\frac{x^2-a^2+a^2}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} + \underbrace{\int -\sqrt{a^2-x^2}\,dx}_{-I} + \int\frac{a^2dx}{\sqrt{a^2-x^2}}$$

$$2I = x\sqrt{a^2-x^2} + \int\frac{a^2dx}{\sqrt{a^2\left(1-\left(\frac{x}{a}\right)^2\right)}} = x\sqrt{a^2-x^2} + \int\frac{a^2\,dx}{|a|\cdot\sqrt{1-\left(\frac{x}{a}\right)^2}} = x\sqrt{a^2-x^2} + \int\frac{a^2dt}{\sqrt{1-t^2}}$$

$$2I = x\sqrt{a^2-x^2} + a^2\cdot\arcsin t + C$$

смена: $\frac{x}{a}=t$, $x=at$, $dx=a\,dt$

$|a|=a$ јер пп. $a>0$

$$\boxed{I = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\arcsin\frac{x}{a}+C}$$

<u>II начин</u>: смена $x=a\sin t$ или $x=a\cos t$ (јер је $1-\sin^2t=\cos^2t$, $1-\cos^2t=\sin^2t$)

$$I = \int\sqrt{a^2-x^2}\,dx = \left(\begin{array}{c} x=a\sin t \\ t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \\ dx = a\cos t\,dt \end{array}\right) = \int\sqrt{a^2-a^2\sin^2t}\;a\cos t\,dt = \int|a|\cdot|\cos t|\,a\cos t\,dt$$

$$\underset{\uparrow}{=} \int a^2\cos^2t\,dt = a^2\cdot\int\frac{1+\cos 2t}{2}\,dt = \frac{a^2}{2}\cdot\left(t+\frac{\sin 2t}{2}\right)+C$$

$\cos t>0$ на $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$a>0$ по пп.

$t=\arcsin\frac{x}{a}$, $\sin 2t = 2\sin t\cos t = 2\cdot\frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}} = \frac{2x}{a^2}\sqrt{a^2-x^2}$

$$I = \frac{a^2}{2}\cdot\left(\arcsin\frac{x}{a}+\frac{x}{a^2}\sqrt{a^2-x^2}\right)+C = \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$$

Домаћи: уз смену $x=a\cos t$, $t\in(0,\pi)$

③ $I = \int \sqrt{x^2-a^2}\,dx = ?$ ⌈опет пр-ало⌋

I начин: парцијалном интеграцијом

$$I = \int \sqrt{x^2-a^2}\,dx = \begin{pmatrix} u=\sqrt{x^2-a^2} & du = \frac{1}{2}\cdot\frac{1}{\sqrt{x^2-a^2}}\cdot 2x\,dx = \frac{x}{\sqrt{x^2-a^2}}\,dx \\ dv = dx & v = x \end{pmatrix}$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - \int\frac{a^2\,dx}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2} - I - \int\frac{a^2\,dx}{|a|\sqrt{\left(\frac{x}{a}\right)^2-1}} = x\sqrt{x^2-a^2} - I - \int\frac{a^2\,dt}{\sqrt{t^2-1}} = x\sqrt{x^2-a^2} - I - a^2\cdot\ln\left|t+\sqrt{t^2-1}\right| + C$$

$\left[\ t=\frac{x}{a},\ dt=\frac{dx}{a}\ \right]$

$$2I = x\sqrt{x^2-a^2} - a^2\ln\left|\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2-1}\right| + C = x\sqrt{x^2-a^2} - a^2\cdot\ln\frac{|x+\sqrt{x^2-a^2}|}{a} + C$$

$$\boxed{I = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C}$$

II начин: смена $x = a\,\mathrm{ch}\,t$ $\left[\ t>0\ \right]$ (домаће: смена $x = a\cdot\frac{1}{\sin t}$, $t\in\left(0,\frac{\pi}{2}\right)$)

$$I = \int\sqrt{x^2-a^2}\,dx = \begin{pmatrix} x = a\,\mathrm{ch}\,t \\ dx = a\,\mathrm{sh}\,t \end{pmatrix} = \int\sqrt{a^2(\mathrm{ch}^2t-1)}\cdot a\,\mathrm{sh}\,t\,dt = \int |a|a\,|\mathrm{sh}\,t|\,\mathrm{sh}\,t\,dt$$

$$\underset{\substack{a>0\\ \mathrm{sh}\,t>0}}{=} \int a^2\mathrm{sh}^2t\,dt = a^2\int\frac{\mathrm{ch}\,2t-1}{2}\,dt = \frac{a^2}{2}\cdot\left(\frac{\mathrm{sh}\,2t}{2} - t\right) + C$$

$$\mathrm{ch}\,2t = \mathrm{ch}^2t + \mathrm{sh}^2t = 1+\mathrm{sh}^2t+\mathrm{sh}^2t = 1+2\,\mathrm{sh}^2t$$

$$\mathrm{sh}^2t = \frac{\mathrm{ch}\,2t-1}{2},\quad \mathrm{sh}\,t = \sqrt{\mathrm{ch}^2t-1}$$

$$\mathrm{sh}\,2t = 2\,\mathrm{sh}\,t\,\mathrm{ch}\,t = 2\cdot\sqrt{\left(\frac{x}{a}\right)^2-1}\cdot\frac{x}{a} = \frac{2x}{a^2}\sqrt{x^2-a^2}$$

$$I = \frac{\cancel{a^2}}{4}\cdot\frac{2x}{\cancel{a^2}}\sqrt{x^2-a^2} - \frac{a^2}{2}\cdot\mathrm{arcch}\,\frac{x}{a} + C = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cdot\mathrm{arcch}\,\frac{x}{a} + C$$

$u = \mathrm{arcch}\,t = ?$

$$\mathrm{ch}\,u = t$$

$$\frac{e^u+e^{-u}}{2} = t \quad /\cdot 2e^u$$

$$e^{2u}+1 = 2e^u t$$

$$(e^u)^2 - 2e^u t + 1 = 0$$

$$e^u_{1,2} = \frac{2t\pm\sqrt{4t^2-4}}{2} = t\pm\sqrt{t^2-1}$$

$$e^u>0 \Rightarrow +$$

$$e^u = t+\sqrt{t^2-1} \qquad u = \ln\left|t+\sqrt{t^2-1}\right| = \ln\left(t+\sqrt{t^2-1}\right)$$

$$I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cdot\ln\left|\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2-1}\right| + C$$

$$\boxed{I = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\frac{x+\sqrt{x^2-a^2}}{a} + C}$$

(4) $I_1 = \int \sin(\ln x)\,dx = ?$

$I_2 = \int \cos(\ln x)\,dx = ?$

$$I_1 = \int \sin(\ln x)\,dx = \begin{pmatrix} u = \sin(\ln x) & du = \cos(\ln x)\cdot\frac{1}{x}dx \\ dv = dx & v = x \end{pmatrix} = x\sin(\ln x) - \underbrace{\int \cos(\ln x)\,dx}_{I_2}$$

$$I_1 + I_2 = x\cdot\sin(\ln x)$$

$$I_2 = \int \cos(\ln x)\,dx = \begin{pmatrix} u = \cos(\ln x) & du = -\sin(\ln x)\cdot\frac{1}{x} \\ dv = dx & v = x \end{pmatrix} = x\cos(\ln x) + \underbrace{\int \sin(\ln x)\,dx}_{I_1}$$

$$I_2 - I_1 = x\cdot\cos(\ln x)$$

$$2I_2 = x\left(\sin(\ln x) + \cos(\ln x)\right) \Rightarrow I_2 = \frac{x}{2}\cdot\left(\cos(\ln x) + \sin(\ln x)\right) + C$$

$$2I_1 = x\left(\sin(\ln x) - \cos(\ln x)\right) \Rightarrow I_1 = \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) + C$$

(5) $I = \int \sqrt{3x^2+2x+1}\,dx = ?$

1. 2. и 3. за $a = 1$:

$$\int \sqrt{x^2-1}\,dx = \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\cdot\ln|x+\sqrt{x^2-1}| + C$$

$$\int \sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|x+\sqrt{x^2+1}| + C$$

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$$

$$I = \int \sqrt{3x^2+2x+1}\,dx = \int \sqrt{(x\sqrt{3})^2 + 2x\sqrt{3}\cdot\frac{1}{\sqrt{3}} + \left(\frac{1}{\sqrt{3}}\right)^2 + \frac{2}{3}}\,dx$$

$$= \int \sqrt{\left(x\sqrt{3}+\frac{1}{\sqrt{3}}\right)^2 + \frac{2}{3}}\,dx = \int \sqrt{\frac{2}{3}\cdot\left(\frac{3}{2}\cdot 3\cdot\left(x+\frac{1}{3}\right)^2 + 1\right)}\,dx$$

$$= \sqrt{\frac{2}{3}}\cdot\int \sqrt{\left(\frac{3x+1}{\sqrt{2}}\right)^2 + 1}\,dx \underset{*}{=} \sqrt{\frac{2}{3}}\cdot\int \sqrt{t^2+1}\,\frac{\sqrt{2}}{3}\,dt = \frac{2}{3\sqrt{3}}\int \sqrt{t^2+1}\,dt$$

смена: $t = \frac{3x+1}{\sqrt{2}}$, $dt = \frac{3}{\sqrt{2}}dx$

$$I = \frac{2\sqrt{3}}{9}\cdot\left(\frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln\left(t+\sqrt{t^2+1}\right)\right) + C$$

$$I = \frac{2\sqrt{3}}{9}\cdot\left(\frac{1}{2}\cdot\frac{3x+1}{\sqrt{2}}\cdot\sqrt{\left(\frac{3x+1}{\sqrt{2}}\right)^2+1} + \frac{1}{2}\ln\left(\frac{3x+1}{\sqrt{2}} + \sqrt{\left(\frac{3x+1}{\sqrt{2}}\right)^2+1}\right)\right) + C$$

(може се још средити наравно)

$$I = \frac{2\sqrt{3}}{9}\cdot\frac{1}{2}\cdot\left(\frac{3x+1}{\sqrt{2}}\cdot\sqrt{\frac{9x^2+6x+3}{2}} + \ln\left(\frac{3x+1}{\sqrt{2}} + \sqrt{\frac{9x^2+6x+3}{2}}\right)\right) + C$$

$$I = \frac{\sqrt{3}}{9}\cdot\left(\frac{3x+1}{2}\sqrt{3}\sqrt{3x^2+2x+1} + \ln\left(\frac{3x+1+\sqrt{3x^2+2x+1}\cdot\sqrt{3}}{\sqrt{2}}\right)\right) + C$$

$$\boxed{I = \frac{3x+1}{6}\sqrt{3x^2+2x+1} + \frac{\sqrt{3}}{9}\ln\frac{3x+1+\sqrt{3}\cdot\sqrt{3x^2+2x+1}}{\sqrt{2}} + C}$$