③ Случајна величина X има густину расподеле $f(x)=ax\sqrt{1-bx}$, $x\in[0,\frac{1}{b}]$ и $f(x)=0$ за $x\notin[0,\frac{1}{b}]$. Ако је $EX=\frac{1}{7}$, одредити константе a, b, дисперзију DX и вероватноћу $P\{X\geq\frac{1}{5}\}$.

$$f(x)=\begin{cases} ax\sqrt{1-bx}, & x\in[0,\frac{1}{b}] \\ 0, & x\notin[0,\frac{1}{b}]\end{cases}$$

$\underline{b\neq 0}$

$f(x)\geq 0$

$\int_{-\infty}^{+\infty} f(x)\,dx=1$

$a\geq 0$

$\int_0^{\frac{1}{b}} ax\sqrt{1-bx}\,dx=1$

$EX=\frac{1}{7}$

$DX=?$

$P\{X\geq\frac{1}{5}\}=?$

$DX=E(X^2)-(EX)^2$

$EX^2=\int_{-\infty}^{\infty}x^2f(x)\,dx=?$

$P\{X\geq\frac{1}{5}\}=1-P\{X<\frac{1}{5}\}=1-F_X(\frac{1}{5})$

$$\int_0^{\frac{1}{b}} ax\sqrt{1-bx}\,dx=1$$

$t=\sqrt{1-bx}$

$t^2=1-bx \qquad d(t^2)=-b\,dx$

$2t\,dt=-b\,dx$

$$\int_1^0 a\cdot\frac{1-t^2}{b}\cdot t\cdot\frac{2t}{-b}\,dt=1$$

$$\frac{2a}{b^2}\int_0^1 t^2(1-t^2)\,dt=1$$

$$\frac{2a}{b^2}\left(\frac{t^3}{3}\Big|_0^1-\frac{t^5}{5}\Big|_0^1\right)=1$$

$$\frac{2a}{b^2}\left(\frac{1}{3}-\frac{1}{5}\right)=1$$

$$\frac{2a}{b^2}\cdot\frac{2}{15}=1$$

$$\boxed{\frac{a}{b^2}=\frac{15}{4}}$$

$EX=\frac{1}{7}$

$$\int_{-\infty}^{\infty}x f(x)\,dx=\frac{1}{7}$$

$$\int_0^{\frac{1}{b}} ax^2\sqrt{1-bx}\,dx=\frac{1}{7} \quad \text{(иста смена } t^2=1-bx\text{)}$$

$$a\int_1^0\left(\frac{1-t^2}{b}\right)^2\cdot t\cdot\frac{2t}{-b}\,dt=\frac{1}{7}$$

$$\frac{2a}{b^3}\int_0^1(1-2t^2+t^4)t^2\,dt=\frac{1}{7}$$

$$\int_0^1(t^2-2t^4+t^6)\,dt=\frac{b^3}{14a}$$

$$\frac{t^3}{3}\Big|_0^1-2\frac{t^5}{5}\Big|_0^1+\frac{t^7}{7}\Big|_0^1=\frac{b^3}{14a}$$

$$\frac{1}{3}-\frac{2}{5}+\frac{1}{7}=\frac{b^3}{14a}$$

$$\frac{35-2\cdot 21+15}{3\cdot 5\cdot 7}=\frac{b^3}{14a}$$

$$\frac{8}{3\cdot 5\cdot 7}=\frac{b^3}{14a} \quad /\cdot 7$$

$$\frac{8}{15}=\frac{b^3}{2a}$$

$$\boxed{15b^3=16a} \qquad \boxed{15b^2=4a}$$

$$\Downarrow \quad 60b^2=15b^3,\ b\neq 0 \Rightarrow \boxed{b=4}$$

$$\boxed{a=60}$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx=\int_0^{\frac{1}{4}} 60\cdot x^3\sqrt{1-4x}\,dx = 60\cdot\int_0^{\frac{1}{4}} x^3\sqrt{1-4x}\,dx$$

$$t^2=1-4x$$

$$2t\,dt=-4dx$$

$$t\,dt=-2dx$$

$$E(X^2)=60\cdot\int_1^0\left(\frac{1-t^2}{4}\right)^3\cdot t\ \frac{t\,dt}{-2}=\frac{\cancel{60}^{15}}{2\cdot 16\cdot \cancel{4}_1}\cdot\int_0^1 t^2(1-t^2)^3\,dt$$

$$=\frac{15}{32}\cdot\int_0^1 t^2(1-3t^2+3t^4-t^6)\,dt=\frac{15}{32}\cdot\int_0^1(t^2-3t^4+3t^6-t^8)\,dt$$

$$=\frac{15}{32}\cdot\left(\frac{t^3}{3}\Big|_0^1-\frac{3t^5}{5}\Big|_0^1+3\frac{t^7}{7}\Big|_0^1-\frac{t^9}{9}\Big|_0^1\right)=\frac{15}{32}\cdot\left(\frac{1}{3}-\frac{3}{5}+\frac{3}{7}-\frac{1}{9}\right)$$

$$=\frac{\cancel{15}}{32}\cdot\frac{5\cdot 7\cdot 3-3\cdot 9\cdot 7+9\cdot 5\cdot 3-5\cdot 7}{\cancel{3}\cdot\cancel{5}\cdot 7\cdot 3}=\frac{5\cdot 21-9\cdot 21+5\cdot(27-7)}{32\cdot 21}=\frac{-21\cdot 4+100}{32\cdot 21}$$

$$=\frac{\cancel{16}^1}{\cancel{32}_2\cdot 21}=\frac{1}{42}$$

$$DX=E(X^2)-(EX)^2=\frac{1}{42}-\left(\frac{1}{7}\right)^2=\frac{1}{42}-\frac{1}{49}=\frac{7}{42\cdot 49}=\frac{1}{42\cdot 7}=\frac{1}{294}\approx 0{,}003$$

$$P\{X \geq \frac{1}{5}\} = 1 - P\{X < \frac{1}{5}\} = 1 - F_X\left(\frac{1}{5}\right) = 1 - 60 \cdot \int_0^{\frac{1}{5}} x\sqrt{1-4x}\,dx$$

$t = \sqrt{1-4x}$

$t^2 = 1-4x$

$2t\,dt = -4dx$

$dx = \frac{-t\,dt}{2}$

$$= 1 - \cancel{60}^{15} \cdot \int_1^{\sqrt{1-\frac{4}{5}}} \frac{1-t^2}{\cancel{4}_1} \cdot t \cdot \frac{t\,dt}{-2} = 1 - \frac{15}{2} \int_{\sqrt{\frac{1}{5}}}^{1} t^2(1-t^2)\,dt$$

$$= 1 - \frac{15}{2} \cdot \int_{\frac{1}{\sqrt{5}}}^{1} (t^2 - t^4)\,dt = 1 - \frac{15}{2}\left(\frac{t^3}{3}\Big|_{\frac{1}{\sqrt{5}}}^{1} - \frac{t^5}{5}\Big|_{\frac{1}{\sqrt{5}}}^{1} \right)$$

$$= 1 - \frac{15}{2} \cdot \left(\frac{1}{3} - \frac{1}{5 \cdot 3\sqrt{5}} - \frac{1}{5} + \frac{1}{5 \cdot 5 \cdot 5\sqrt{5}} \right) = 1 - \frac{\cancel{15}}{2} \cdot \frac{125\sqrt{5} - 25 - 75\sqrt{5} + 3}{\cancel{3 \cdot 5} \cdot 25\sqrt{5}}$$

$$= 1 - \frac{50\sqrt{5} - 22}{50\sqrt{5}} = \frac{22}{50\sqrt{5}} = \frac{11}{25\sqrt{5}}$$

$$\approx 0{,}197$$