Додатак: задаци са испитних рокова (вероватноћа)

(1) Случајна величина X има густину расподеле $f(x)=\begin{cases}(ax^2+x+c)e^{-2x}, & x\geq 0\\ 0, & x<0\end{cases}$

где су $a,c\in\mathbb{R}$ и важи $EX=1$.

Наћи константе a,c и дисперзију DX. Одредити $P\{X\leq \frac{1}{2}\}$.

$$f(x)=\begin{cases}(ax^2+x+c)e^{-2x}, & x\geq 0\\ 0, & x<0\end{cases}$$

$$\frac{\begin{array}{l} a,c\in\mathbb{R}\\ EX=1\end{array}}{\begin{array}{l} DX=?\\ P\{X\leq\frac{1}{2}\}=?\end{array}}$$

Да би f била густина расподеле мора да важи $f(x)\geq 0$ и $\int_{-\infty}^{\infty} f(x)\,dx=1$.

Услов $EX=1$ даје $\int_{-\infty}^{\infty} x f(x)\,dx=1$.

Из свих ових услова налазимо a и c. Затим, $DX=E(X^2)-(EX)^2$

и $E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx$.

$P\{X\leq\frac{1}{2}\}=F_X(\frac{1}{2})$ где је F ф-ја расподеле

$$F_X(x)=\int_{-\infty}^{x} f(t)\,dt.$$

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_0^{+\infty}(ax^2+x+c)e^{-2x}\,dx=a\cdot\int_0^{+\infty}x^2e^{-2x}\,dx+\int_0^{+\infty}xe^{-2x}\,dx+c\cdot\int_0^{+\infty}e^{-2x}\,dx$$

$$\int_0^{+\infty}e^{-2x}\,dx=\frac{e^{-2x}}{-2}\Big|_0^{+\infty}=0+\frac{1}{2}e^{-2\cdot 0}=\frac{1}{2}$$

(рачунамо сваки интеграл посебно, јер ће нам требати и у даљем раду, па због краћег записа)

$$\int_0^{+\infty}xe^{-2x}\,dx=\begin{pmatrix}u=x & du=dx\\ dv=e^{-2x}dx & v=\frac{e^{-2x}}{-2}\end{pmatrix}=-\frac{1}{2}xe^{-2x}\Big|_0^{+\infty}+\frac{1}{2}\int_0^{+\infty}e^{-2x}\,dx$$

$$=\overbrace{\lim_{x\to\infty}\frac{x\cdot(-1)}{2e^{2x}}}^{=0}+\overbrace{\frac{1}{2}\cdot 0e^{-2\cdot 0}}^{=0}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

$$\int_0^{+\infty} x^2 e^{-2x}dx = \begin{pmatrix} u=x^2 & du=2x\,dx \\ dv=e^{-2x}dx & v=\frac{e^{-2x}}{-2}\end{pmatrix} = -\frac{1}{2}x^2e^{-2x}\Big|_0^{+\infty} + \frac{1}{2}\int_0^{+\infty} e^{-2x}\, 2x\,dx$$

$$= \underbrace{\lim_{x\to+\infty} \frac{-x^2}{2e^{2x}}}_{=0} + \underbrace{\frac{1}{2}\cdot 0^2\cdot e^{-2\cdot 0}}_{=0} + \frac{1}{2}\cdot 2\cdot \int_0^{+\infty} x e^{-2x}dx = \int_0^{+\infty} x e^{-2x}dx = \frac{1}{4}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)\,dx = a\cdot\frac{1}{4} + \frac{1}{4} + c\cdot\frac{1}{2} = \frac{a+1+2c}{4} = 1$$

$$\boxed{a+2c=3}$$

$$EX = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{+\infty} x\cdot(ax^2+x+c)e^{-2x}dx$$

$$= a\cdot\int_0^{+\infty} x^3e^{-2x}dx + \underbrace{\int_0^{+\infty} x^2e^{-2x}dx}_{=\frac{1}{4}} + c\cdot\underbrace{\int_0^{+\infty} xe^{-2x}dx}_{=\frac{1}{4}}$$

$$\int_0^{+\infty} x^3e^{-2x}dx = \begin{pmatrix} u=x^3 & du=3x^2dx \\ dv=e^{-2x}dx & v=\frac{e^{-2x}}{-2}\end{pmatrix} = \underbrace{-\frac{1}{2}x^3e^{-2x}\Big|_0^{+\infty}}_{=0} + \frac{1}{2}\int_0^{+\infty} e^{-2x}\, 3x^2dx$$

$$= \frac{3}{2}\cdot\int_0^{+\infty} e^{-2x}\cdot x^2dx = \frac{3}{2}\cdot\frac{1}{4} = \frac{3}{8}$$

$$EX = 1 \Rightarrow \quad \frac{3}{8}a + \frac{1}{4} + \frac{1}{4}c = 1 \quad /\cdot 8$$

$$3a+2+2c=8$$

$$\boxed{3a+2c=6}$$

$$3a+2c=6$$
$$a+2c=3$$

$$2a=3$$

$$\boxed{a=\frac{3}{2}}$$

$$2c = 3-a = 3-\frac{3}{2} = \frac{3}{2}$$

$$\boxed{c=\frac{3}{4}}$$

Одредимо сада $E(X^2)$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{+\infty} x^2\cdot\left(\frac{3}{2}x^2+x+\frac{3}{4}\right)e^{-2x}dx$$

$$= \frac{3}{2}\int_0^{+\infty} x^4e^{-2x}dx + \underbrace{\int_0^{+\infty} x^3e^{-2x}dx}_{=\frac{3}{8}} + \frac{3}{4}\underbrace{\int_0^{+\infty} x^2e^{-2x}dx}_{\frac{1}{4}}$$

$$\int_0^{+\infty} x^4e^{-2x}dx = \begin{pmatrix} u=x^4 & du=4x^3dx \\ dv=e^{-2x}dx & v=\frac{e^{-2x}}{-2}\end{pmatrix}$$

$$= \underbrace{-\frac{1}{2}x^4e^{-2x}\Big|_0^{+\infty}}_{=0} + \frac{1}{2}\int_0^{+\infty} 4x^3e^{-2x}dx$$

$$= 2\cdot\int_0^{+\infty} x^3e^{-2x}dx = 2\cdot\frac{3}{8} = \frac{3}{4}$$

$$\Rightarrow \quad E(X^2) = \frac{3}{2}\cdot\frac{3}{4} + \frac{3}{8} + \frac{3}{4}\cdot\frac{1}{4} = \frac{9}{8} + \frac{3}{8} + \frac{3}{16} = \frac{18+6+3}{16} = \frac{27}{16}$$

$$\Rightarrow \quad DX = E(X^2) - (EX)^2 = \frac{27}{16} - 1 = \frac{11}{16}$$

$$\boxed{DX = \frac{11}{16}}$$

$$P\{X \le \tfrac{1}{2}\} = F_X(\tfrac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} f(t)\,dt = \int_0^{\frac{1}{2}} \left(\tfrac{3}{2}x^2 + x + \tfrac{3}{4}\right) e^{-2x}\,dx$$

$$= \frac{3}{2}\cdot\int_0^{\frac{1}{2}} x^2 e^{-2x}\,dx + \int_0^{\frac{1}{2}} x e^{-2x}\,dx + \frac{3}{4}\cdot\int_0^{\frac{1}{2}} e^{-2x}\,dx$$

$u = x^2$ $du = 2x\,dx$ $dv = e^{-2x}dx$ $v = \frac{e^{-2x}}{-2}$ $u = x$ $du = dx$ $dv = e^{-2x}dx$ $v = \frac{e^{-2x}}{-2}$

$$= \frac{3}{2}\cdot\left(-\frac{1}{2}x^2 e^{-2x}\Big|_0^{\frac{1}{2}} + \frac{1}{2}\cdot 2\int_0^{\frac{1}{2}} x e^{-2x}\,dx\right) + \left(-\frac{1}{2}x e^{-2x}\Big|_0^{\frac{1}{2}} + \frac{1}{2}\cdot\int_0^{\frac{1}{2}} e^{-2x}\,dx\right) + \frac{3}{4}\cdot\frac{e^{-2x}}{-2}\Big|_0^{\frac{1}{2}}$$

$$= \frac{3}{2}\cdot\left(-\frac{1}{2}\cdot\frac{1}{4}e^{-1} + \int_0^{\frac{1}{2}} x e^{-2x}\,dx\right) + \left(-\frac{1}{2}\cdot\frac{1}{2}e^{-1} + \frac{1}{2}\cdot\underbrace{\frac{e^{-2x}}{-2}\Big|_0^{\frac{1}{2}}}_{\frac{1-e^{-1}}{2}}\right) + \frac{3}{4}\cdot\left(-\frac{1}{2}\right)\cdot(e^{-1} - 1)$$

$$= \frac{3}{2}\left(\frac{-1}{8e} + \int_0^{\frac{1}{2}} x e^{-2x}\,dx\right) + \left(\frac{-1}{4e} + \frac{e-1}{4e}\right) - \frac{3}{8}\cdot\left(\frac{1}{e} - 1\right)$$

$$= \frac{3}{2}\left(\frac{-1}{8e} - \frac{1}{4e} + \frac{e-1}{4e}\right) + \frac{e-2}{4e} - \frac{3-3e}{8e}$$

$$= \frac{3}{2}\cdot\frac{-1-2+2e-2}{8e} + \frac{2e-4-3+3e}{8e} = \frac{6e-15+10e-14}{16e} = \frac{16e-29}{16e}$$

$$\boxed{P\{X \le \tfrac{1}{2}\} = 1 - \frac{29}{16e}} \approx 0{,}33$$

② Случајна величина X има густину расподеле $f(x)=ae^{-|x-b|}$, $x\in\mathbb{R}$ где су $a,b\in\mathbb{R}$ и важи $EX=5$. Наћи константе a, b, дисперзију DX и вероватноћу $P\{X\geq 3\}$.

$$f(x)=ae^{-|x-b|}$$

$$EX=5$$

$$DX=?$$

$$P\{X\geq 3\}=?$$

$$f(x)\geq 0 \Rightarrow \underline{\underline{a\geq 0}}$$

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$$|x-b|=\begin{cases} x-b, & x\geq b \\ b-x, & x<b \end{cases}$$

$$\int_{-\infty}^{\infty} ae^{-|x-b|}\,dx=1$$

$$\int_{-\infty}^{b} ae^{-(b-x)}\,dx+\int_{b}^{+\infty} ae^{-(x-b)}\,dx=1$$

$$\int_{-\infty}^{b} ae^{x-b}\,dx+\int_{b}^{+\infty} ae^{b-x}\,dx=1$$

$t=x-b$ $\qquad$ $t=b-x$

$dt=dx$ $\qquad$ $dt=-dx$

$$\int_{-\infty}^{0} a\cdot e^{t}\,dt+\int_{0}^{-\infty} ae^{t}(-dt)=1 \qquad \left(=\int_{-\infty}^{0} ae^{t}\,dt\right)$$

$$a\cdot e^{t}\Big|_{-\infty}^{0}+a\cdot e^{t}\Big|_{-\infty}^{0}=1$$

$$a-\underbrace{\lim_{t\to-\infty} ae^{t}}_{=0}+a-\underbrace{\lim_{t\to-\infty} ae^{t}}_{=0}=1$$

$$\boxed{2a=1}\Rightarrow\boxed{a=\frac{1}{2}}$$

$$EX=5=\int_{-\infty}^{\infty} x f(x)\,dx=\int_{-\infty}^{b} axe^{-(b-x)}\,dx+\int_{b}^{+\infty} axe^{-(x-b)}\,dx=\frac{1}{2}\cdot\left(\int_{-\infty}^{b} xe^{x-b}\,dx+\int_{b}^{+\infty} xe^{b-x}\,dx\right)$$

$$\int_{-\infty}^{b} xe^{x-b}\,dx+\int_{b}^{+\infty} xe^{b-x}\,dx=10$$

$t=x-b$ $\qquad$ $t=b-x$

$dt=dx$ $\qquad$ $dt=-dx$

$$\int_{-\infty}^{0}(t+b)e^{t}\,dt+\int_{0}^{-\infty}(b-t)e^{t}(-dt)=10$$

$$\int_{-\infty}^{0}(\cancel{t}+b+b-\cancel{t})e^{t}\,dt=10$$

$$2b\cdot\int_{-\infty}^{0} e^{t}\,dt=10$$

$$2b\cdot e^{t}\Big|_{-\infty}^{0}=10$$

$$2b=10$$

$$\boxed{b=5}$$

$$E(X^2)=a\int_{-\infty}^{\infty} x^2 e^{-|x-6|}\,dx = \frac{1}{2}\cdot \int_{-\infty}^{\infty} x^2 e^{-|x-5|}\,dx$$

$$= \frac{1}{2}\cdot\left(\int_{-\infty}^{5} x^2 e^{x-5}\,dx + \int_{5}^{+\infty} x^2 e^{5-x}\,dx \right) = \frac{1}{2}\cdot\left(\int_{-\infty}^{0} (t+5)^2\cdot e^{t}\,dt + \int_{0}^{-\infty} (5-t)^2 e^{t}(-dt)\right)$$

$$\ulcorner\, t = x-5 \qquad dt = dx \,\lrcorner \qquad\qquad \ulcorner\, t=5-x \qquad dt=-dx\,\lrcorner$$

$$= \frac{1}{2}\cdot\left(\int_{-\infty}^{0} (t+5)^2 e^{t}\,dt + \int_{-\infty}^{0} (5-t)^2 e^{t}\,dt \right)$$

$$= \frac{1}{2}\cdot \int_{-\infty}^{0} \left(t^2 + \cancel{10t} + 25 + 25 - \cancel{10t} + t^2\right) e^{t}\,dt$$

$$= \int_{-\infty}^{0} (t^2+25)e^{t}\,dt = \int_{-\infty}^{0} t^2 e^{t}\,dt + \underbrace{25e^{t}\Big|_{-\infty}^{0}}_{=25}$$

$$= t^2e^{t}\Big|_{-\infty}^{0} - 2\int_{-\infty}^{0} t e^{t}\,dt + 25 \qquad \ulcorner\, u=t^2 \quad du = 2t\,dt;\quad dv = e^{t}dt \quad v=e^{t}\,\lrcorner$$

$$= 25 - 2\cdot\int_{-\infty}^{0} t e^{t}\,dt \qquad \left(u=t \quad du=dt;\quad dv=e^{t}dt \quad v=e^{t} \right)$$

$$= 25 - 2\cdot\left(\underbrace{te^{t}\Big|_{-\infty}^{0}}_{=0} - \int_{-\infty}^{0} e^{t}\,dt \right) = 25 + 2\cdot e^{t}\Big|_{-\infty}^{0} = 27$$

$$\Rightarrow\ DX = E(X^2) - (EX)^2 = 27 - 5^2 = 27-25 = 2$$

$$\boxed{DX = 2}$$

$$P\{X \geq 3\} = 1 - P\{X<3\} = 1 - F_X(3) = 1 - \int_{-\infty}^{3} f(t)\,dt$$

$$= 1 - \int_{-\infty}^{3} \frac{1}{2} e^{-|t-5|}\,dt = 1 - \int_{-\infty}^{3}\frac{1}{2}e^{t-5}\,dt = 1 - \frac{1}{2}\, e^{t-5}\Big|_{-\infty}^{3}$$

$$= 1 - \frac{1}{2}\left(e^{3-5} - \underbrace{\lim_{t\to-\infty} e^{t-5}}_{=0}\right) = 1 - \frac{1}{2e^2} \approx 0{,}93 .$$