<u>Лема</u> Ако је $f(x)$ непрекидна функција и $\alpha_i > 0$ тада ће постојати ξ такво да је

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) + \dots + \alpha_n f(x_n) = (\alpha_1 + \dots + \alpha_n) f(\xi)$$

где ξ припада најмањем интервалу којем припадају сви x_i.

<u>Доказ</u>

$$\frac{1}{\alpha_1 + \dots + \alpha_n} (\alpha_1 f(x_1) + \dots + \alpha_n f(x_n)) =$$

$$\frac{\alpha_1}{\alpha_1 + \dots + \alpha_n} f(x_1) + \dots + \frac{\alpha_n}{\alpha_1 + \dots + \alpha_n} f(x_n) = \beta_1 f(x_1) + \dots + \beta_n f(x_n)$$

$$\beta_i > 0 \quad , \quad \sum_1^n \beta_i = 1$$

$A = \beta_1 f(x_1) + \dots + \beta_n f(x_n)$ је конвексна комбинација вредности $f(x_1), \dots, f(x_n)$ и припада најмањем интервалу који садржи те вредности.

Пошто је $f(x)$ непрекидна, узима све међувредности па мора постојати ξ такво да је $f(\xi) = A$

$$f(\xi) = A = \beta_1 f(x_1) + \dots + \beta_n f(x_n) = \frac{1}{(\alpha_1 + \dots + \alpha_n)} (\alpha_1 f(x_1) + \dots + f(x_n))$$

$$f'(x_0) \approx \frac{f(x_1)-f(x_0)}{h}$$

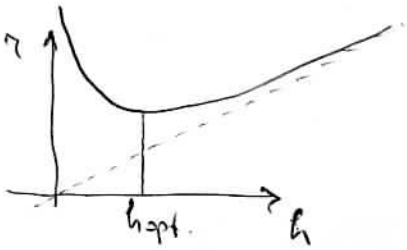
$$x_1 = x_0 + h \qquad f(x_1) = f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(\xi)$$

$$\frac{f(x_1)-f(x_0)}{h} = \frac{1}{h}\left[f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(\xi) - f(x_0) \right]$$

$$= f'(x_0) + \frac{h}{2} f''(\xi)$$

$$f'(x_0) = \frac{f(x_1)-f(x_0)}{h} - \frac{h}{2} f''(\xi) \quad \Rightarrow \quad R_M \le \frac{h}{2} M_2$$

$$R_R \le \frac{2\varepsilon}{h}$$



$$R \le R_1 + R_2 = r(h) = \frac{h}{2} M_2 + \frac{2\varepsilon}{h} \rightarrow \min_h$$

$$r'(h) = \frac{M_2}{2} - \frac{2\varepsilon}{h^2} = 0 \Rightarrow h_{opt} = \sqrt{\frac{4\varepsilon}{M_2}}$$

Ropt = 2*sqrt(M2*eps)

$$f''(x_0) \approx \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2}$$

$$\frac{1}{h^2}\left[f(x_0-h) - 2f(x_0) + f(x_0+h) \right] = \frac{1}{h^2}\Big[f(x_0) - h f'(x_0) + \frac{h^2}{2} f''(x_0) - \frac{h^3}{6} f'''(x_0) + \frac{h^4}{24} f^{IV}(\xi_1)$$

$$- 2 f(x_0)$$

$$+ f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f'''(x_0) + \frac{h^4}{24} f^{IV}(\xi_2) \Big]$$

$$= f''(x_0) + \frac{h^2}{24}\left(f^{IV}(\xi_1) + f^{IV}(\xi_2) \right)$$

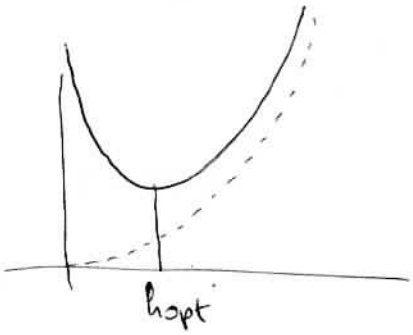
$$\left(\alpha_1 f(\xi_1) + \alpha_2 f(\xi_2) = (\alpha_1 + \alpha_2) f(\xi_3) \right)$$

$$\alpha_1, \alpha_2 > 0$$

$$= f''(x_0) + \frac{h^2}{12} f^{IV}(\xi)$$

$$f''(x_0) = \frac{f(x_0-h) - 2f(x_0) + f(x_0+h)}{h^2} - \frac{h^2}{12} f^{IV}(\xi) \quad \Rightarrow \quad R_M \le \frac{h^2}{12} M_4$$

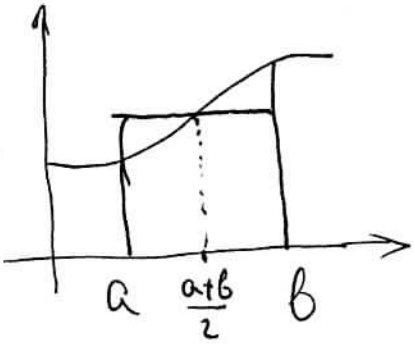
$$R_R \le \frac{4\varepsilon}{h^2}$$



$$R \le R_1 + R_2 = r(h) = \frac{h^2}{12} M_4 + \frac{4\varepsilon}{h^2} \rightarrow \min_h$$

$$r'(h) = \frac{h}{6} M_4 - \frac{8\varepsilon}{h^3} = 0 \Rightarrow h_{opt.} = \sqrt[4]{\frac{48\varepsilon}{M_4}}$$

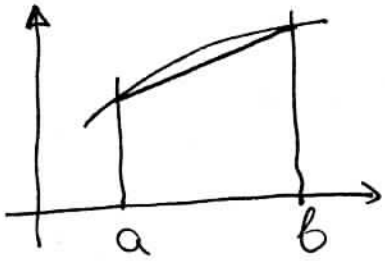
Ropt = 4/3*sqrt(3M4*eps)



$$\int_a^b f(x)dx \approx c\cdot f\left(\frac{a+b}{2}\right)$$

$$f(x)\equiv 1 \Rightarrow \int_a^b dx = (b-a) = c$$

$$S_0(f) = (b-a)\, f\left(\frac{a+b}{2}\right)$$



$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

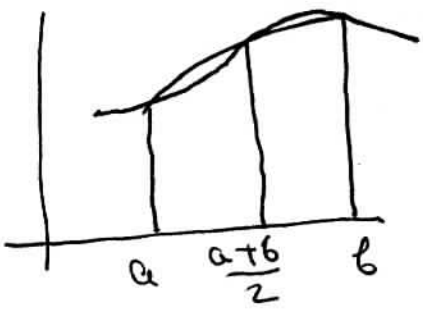
$$f(x)\equiv 1 \Rightarrow (b-a) = c_1 + c_2$$

$$f(x)=x \Rightarrow \frac{1}{2}(b^2-a^2) = c_1 a + c_2 b$$

симетрија $\Rightarrow c_1 = c_2$

$$c_1 = c_2 = \frac{1}{2}(b-a)$$

$$S_1(f) = \frac{b-a}{2}\left(f(a)+f(b)\right)$$



$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

$$f(x)\equiv 1 \Rightarrow (b-a) = c_1 + c_2 + c_3$$

$$f(x)=x \Rightarrow \frac{1}{2}(b^2-a^2) = c_1 a + c_2 \frac{a+b}{2} + c_3 b$$

$$f(x)=x^2 \Rightarrow \frac{1}{3}(b^3-a^3) = c_1 a^2 + c_2\left(\frac{a+b}{2}\right)^2 + c_3 b^2$$

симетрија $\Rightarrow c_1 = c_3$

$$c_1 = c_3 = \frac{b-a}{6} \qquad c_2 = \frac{4}{3}(b-a)$$

$$S_2(f) = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

$$x = \frac{a+b}{2} + \frac{b-a}{2}t \qquad [a,b] \to [-1,1]$$

$$\int_a^b L_n(x)dx = \frac{b-a}{2}\int_{-1}^{1} \overline{L}_n(t)dt$$

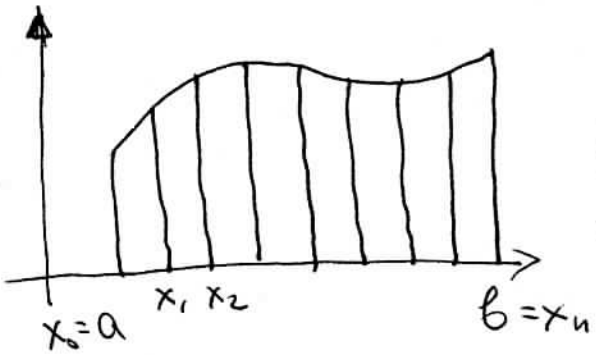
$$f(t)\equiv 1: \quad \int_{-1}^{1} dt = 2 = c_1 + c_2 + c_3$$

$$f(t)=t: \quad \int_{-1}^{1} t\,dt = 0 = c_1(-1) + c_2\cdot 0 + c_3\cdot 1$$

$$f(t)=t^2: \quad \int_{-1}^{1} t^2 dt = \frac{2}{3} = c_1\cdot 1^2 + c_2\cdot 0 + c_3\cdot 1^2$$

$$c_1 = c_3 = \frac{1}{3} \qquad c_2 = \frac{4}{3}$$

$$S_2(*) \quad \frac{b-a}{2}\cdot\frac{1}{3}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$



$$h = \frac{b-a}{n}$$

$$\int_a^b = \int_{x_0}^{x_1} + \int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n}$$

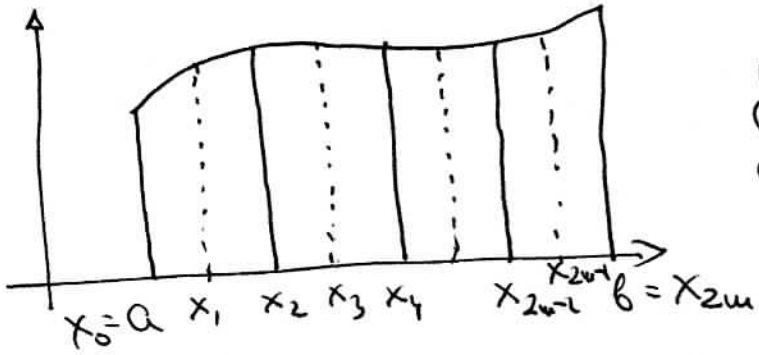
$$S_0^h(f) = (x_1 - x_0) f(x_0 + \tfrac{h}{2}) + \cdots + (x_n - x_{n-1}) f(x_{n-1} + \tfrac{h}{2})$$

$$= h\,(f_{1/2} + \cdots + f_{i-1/2} + \cdots + f_{n-1/2})$$

$$S_1^h(f) = \frac{x_1 - x_0}{2}(f_0 + f_1) + \frac{x_2 - x_1}{2}(f_1 + f_2) + \cdots + \frac{x_n - x_{n-1}}{2}(f_{n-1} + f_n)$$

$$= h\left(\tfrac{1}{2} f_0 + (f_1 + \cdots + f_{n-1}) + \tfrac{1}{2} f_n\right)$$

$$n = 2m \qquad h = \frac{b-a}{2m}$$



$$\int_a^b = \int_{x_0}^{x_2} + \int_{x_2}^{x_4} + \cdots + \int_{x_{2m-2}}^{x_{2m}}$$

$$S_2^h(f) = \frac{x_2 - x_0}{6}(f_0 + 4f_1 + f_2) + \frac{x_4 - x_2}{6}(f_2 + 4f_3 + f_4) + \cdots +$$

$$+ \frac{x_{2m} - x_{2m-2}}{6}(f_{2m-2} + 4f_{2m-1} + f_{2m})$$

$$= \frac{h}{3}\left[f_0 + f_{2m} + 4(f_1 + f_3 + \cdots + f_{2m-1}) + 2(f_2 + f_4 + \cdots + f_{2m-2})\right]$$