

$\lim_{R \rightarrow \infty} \int_{\gamma_R} f(z) e^{iz^2} dz = 0$ на основу Морданова леме 3

$$\left(\lim_{\substack{|z| \rightarrow \infty \\ z \in \mathbb{H}}} f(z) = \lim_{\substack{|z| \rightarrow \infty \\ z \in \mathbb{H}}} \frac{1}{z/(z^2+1)^2} = 0, f \text{ је непрекидна на } H_\delta = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0, |z| > \delta\} \right)$$

$\lim_{\epsilon \rightarrow 0^+} \int_{\ell_\epsilon^-} f(z) e^{iz^2} dz = -i \cdot A \cdot (\pi - 0), A = \lim_{z \rightarrow 0} z \cdot f(z) e^{iz^2}$

Морданова
лема 1

$$A = \lim_{z \rightarrow 0} z \cdot \frac{e^{iz^2}}{z/(z^2+1)^2} = 1$$

$$\int_{\ell_1} f(z) e^{iz^2} dz = \int_{-R}^{-\epsilon} f(t) e^{iat} dt = \int_{R}^{\epsilon} f(-u) e^{-iau} du = \int_{-\epsilon}^R f(-u) e^{-iau} du$$

$$\int_{\ell_2} f(z) e^{iz^2} dz = \int_{\epsilon}^R f(t) e^{iat} dt$$

$$f(z) = \frac{1}{z/(z^2+1)^2} \quad f(t) = -f(-t) \quad \text{нечарка је тја}$$

$$\begin{aligned} \int_{\ell_1} f(z) e^{iz^2} dz + \int_{\ell_2} f(z) e^{iz^2} dz &= \int_{-\epsilon}^R f(t) e^{iat} dt + \int_{\epsilon}^R (-f(t)) e^{-iat} dt \\ &= \int_{\epsilon}^R f(t) (\cos at + i \sin at - \cos at - i \sin at) dt \\ &= 2i \int_{\epsilon}^R f(t) \sin at dt \end{aligned}$$

$$\Rightarrow S = S(R, \epsilon) \quad / \lim_{\substack{R \rightarrow \infty \\ \epsilon \rightarrow 0}}$$

$$S = 0 + 2i \cdot \int_0^{+\infty} f(t) \sin at dt - i A \pi = 2i \cdot \underbrace{\int_0^{+\infty} \frac{\sin at}{t/(t^2+1)^2} dt}_{I} - i \pi$$

$$2i I - i \pi = \frac{-i \pi (a+2) e^{-a}}{2} \quad /: 2i \quad I$$

$$I = \frac{\pi}{2} + \frac{-\pi (a+2) e^{-a}}{4} = \frac{\pi (2 - (a+2) e^{-a})}{4}$$

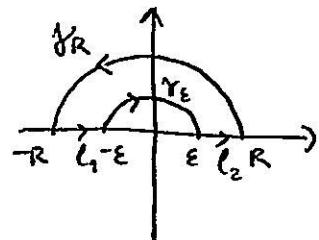
$$\textcircled{5} \quad \text{Израчунати: } I = \int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx$$

$$I = \int_0^\infty \frac{\sin^2 x}{x^2} dx = \int_0^\infty \frac{1 - \cos 2x}{2x^2} dx = \frac{1}{2} \cdot \int_0^\infty \frac{1 - \cos 2x}{x^2} dx$$

чије је још класичан
тип, али разни
се начини начин!

Уочимо да је $g(z) = \frac{1 - e^{2iz}}{z^2}$, $\Gamma_{R,\varepsilon} = \gamma_R + \ell_1 + \gamma_\varepsilon + \ell_2$
 $0 < \varepsilon < R$

$$\gamma_R: z = Re^{it}, t \in [0, \pi]$$



$$\ell_1: z = t, t \in [-R, -\varepsilon]$$

$$\gamma_\varepsilon: z = \varepsilon e^{it}, t \in [0, \pi] \quad (\text{ам ог } \pi \text{ као})$$

$$\ell_2: z = t, t \in [\varepsilon, R].$$

$\int_{\Gamma_{R,\varepsilon}} g(z) dz = 0$ јер да није сингуларитет у $int \Gamma_{R,\varepsilon}$
(на основу Кошијеве Т, g је холом. на
окolini $\Gamma_{R,\varepsilon}$)

(може се речи и на оставу Т-о осцилација,
суме резултата је 0, јер да није)

$$\int_{\gamma_R} g(z) dz + \int_{\ell_1} g(z) dz + \int_{\gamma_\varepsilon} g(z) dz + \int_{\ell_2} g(z) dz = 0$$

$$\int_{\ell_1} g(z) dz + \int_{\ell_2} g(z) dz = \int_{-R}^{-\varepsilon} g(t) dt + \int_{\varepsilon}^R g(t) dt = \int_{-\varepsilon}^R g(-t) dt + \int_{\varepsilon}^R g(t) dt$$

$$= \int_{-\varepsilon}^R (g(t) + g(-t)) dt = \int_{-\varepsilon}^R \left(\frac{1 - e^{2it}}{t^2} + \frac{1 - e^{-2it}}{t^2} \right) dt$$

$$= \int_{-\varepsilon}^R \frac{2 - \cos 2t - i \sin 2t - \cos 2t + i \sin 2t}{t^2} dt = \int_{-\varepsilon}^R \frac{2 - 2 \cos 2t}{t^2} dt$$

$$\int_{\gamma_R} g(z) dz + \int_{\gamma_\varepsilon} g(z) dz + \int_{-\varepsilon}^R \frac{2 - 2 \cos 2t}{t^2} dt = 0 \quad / \lim_{R \rightarrow \infty} \quad (*)$$

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} g(z) dz = \lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{1 - e^{2iz}}{z^2} dz = \lim_{R \rightarrow \infty} \left(\int_{\gamma_R} \frac{dz}{z^2} - \int_{\gamma_R} \frac{e^{2iz}}{z^2} dz \right)$$

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{dz}{z^2} = i \cdot (\pi - 0) \cdot A, A = \lim_{z \rightarrow \infty} \frac{1}{z^2} = 0 \quad (\text{Норманова лема 2})$$

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{e^{2iz}}{z^2} dz = 0 \quad \begin{matrix} \text{согласно} \\ \text{Норманова лема 3} \end{matrix} \left(\lim_{z \rightarrow \infty} \frac{1}{z^2} = 0, \frac{1}{z^2} \text{ непр. на } \{z \in \mathbb{C} : |m z| > 1, |z| > \varepsilon_1\} \right)$$

$$\Rightarrow \lim_{R \rightarrow \infty} \int_{\gamma_R} g(z) dz = 0$$

$$(*) \Rightarrow 0 = \int_{\gamma_\varepsilon^-} g(z) dz + \int_{\varepsilon}^{+\infty} \frac{2 - 2 \cos t}{t^2} dt \quad / \lim_{\varepsilon \rightarrow 0^+}$$

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\gamma_\varepsilon^-} g(z) dz = \lim_{\varepsilon \rightarrow 0^+} \int_{\gamma_\varepsilon^-} \frac{1 - e^{2iz}}{z^2} dz = -i \cdot (\pi - 0) \cdot B$$

$$B = \lim_{z \rightarrow 0} z \cdot g(z) = \lim_{z \rightarrow 0} z \cdot \frac{1 - e^{2iz}}{z^2}$$

$$B = \lim_{z \rightarrow 0} \frac{1 - e^{2iz}}{z} = \lim_{z \rightarrow 0} \frac{1 - (1 + 2iz + \frac{1}{2} \cdot (2iz)^2 + \dots)}{z}$$

$$B = \lim_{z \rightarrow 0} \frac{-2iz + \frac{1}{2} \cdot 4z^2 + \dots}{z} = -2i$$

Норманова лема 1

$$\Rightarrow \lim_{\varepsilon \rightarrow 0^+} \int_{\gamma_\varepsilon^-} g(z) dz = -i \cdot \pi \cdot (-2i) = 2\pi$$

$$\int_0^{+\infty} \frac{2 - 2 \cos t}{t^2} dt = -(-2\pi) = 2\pi$$

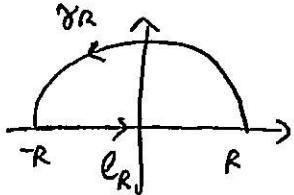
$$\Rightarrow \int_0^{+\infty} \frac{1 - \cos t}{t^2} dt = \pi \Rightarrow \boxed{I = \frac{1}{2} \cdot \pi}$$

$$\text{Изл.} \quad \boxed{\int_0^{+\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}}$$

⑥ Израчунати: $I = \int_0^{+\infty} \frac{\cos nx}{x^4 + 1} dx$, $n \in \mathbb{N}$.

$$I = \operatorname{Re} \int_0^{+\infty} \frac{e^{inx}}{x^4 + 1} dx \quad \left(\text{У интеграл је тип 3, тј. коријесов интеграл} \right)$$

$$f(z) = \frac{1}{z^4 + 1}$$



$$\Gamma_R = \gamma_R + l_R \quad \gamma_R: z = Re^{it}, t \in [0, \pi] \\ l_R: z = t, t \in [-R, R]$$

$$\int_{\Gamma_R} f(z) e^{inz} dz = 2\pi i \cdot \sum_{k=0}^m \operatorname{Res}(f(z) e^{inz}, z_k)$$

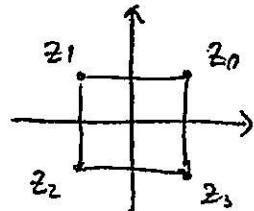
(z_0, \dots, z_m сингуларни чланови интеграла)

Решимо јаку: $z^4 + 1 = 0$

$$z^4 = -1 = e^{i\pi}$$

$$z_k = e^{i \frac{\pi + 2k\pi}{4}}$$

$$z_0 = e^{i \frac{\pi}{4}}, z_1 = e^{i \frac{3\pi}{4}}, z_2 = e^{i \frac{5\pi}{4}}, z_3 = e^{i \frac{7\pi}{4}}$$



Сингуларноста функција $f(z) e^{inz}$ су z_0, z_1, z_2, z_3 , а

у врхују посматране се налазе z_0 и z_1 (за $R > 1$)

$$\Rightarrow \int_{\Gamma_R} f(z) e^{inz} dz = 2\pi i \cdot \underbrace{\left(\operatorname{Res}(f(z) e^{inz}, z_0) + \operatorname{Res}(f(z) e^{inz}, z_1) \right)}_{S}$$

$$\int_{\gamma_R} f(z) e^{inz} dz + \int_{l_R} f(z) e^{inz} dz = 2\pi i \cdot S, R > 1$$

$\lim_{R \rightarrow \infty} \int_{\gamma_R} f(z) e^{inz} dz = 0$ на основу Шорданове леме 3

$$\left(\lim_{\substack{z \rightarrow \infty \\ z \in \mathbb{H}}} f(z) = \lim_{\substack{z \rightarrow \infty \\ z \in \mathbb{H}}} \frac{1}{z^4 + 1} = 0 \right)$$

$$2\pi i \cdot S = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{1}{t^4 + 1} \cdot e^{int} dt$$

f је непр. на $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0, |z| > 2\}$

Ваше утверждение:

$$\int_{-\infty}^{\infty} \frac{e^{int}}{t^4+1} dt = 2\pi i \cdot S$$

Приложение:

f, g холоморфные на $D(z_0, r)$

$f(z_0) \neq 0, g(z_0) = 0, g'(z_0) \neq 0$

$$\Rightarrow \text{Res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)}$$

$$z_0 = e^{i\frac{\pi}{4}}$$

$$\text{Res}\left(\frac{e^{int}}{z^4+1}, e^{i\frac{\pi}{4}}\right) = \frac{e^{in \cdot z_0}}{4z_0^3} = \frac{e^{in \cdot (\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})}}{4 \cdot e^{i\frac{3\pi}{4}}} = \frac{e^{-n\frac{\sqrt{2}}{2} + i\frac{n\sqrt{2}}{2}}}{4 \cdot (\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})}$$

$f(z) = e^{int}$
 $g(z) = z^4 + 1$
 $g'(z) = 4z^3$ ТБПУСКЕ

$$\text{Res}\left(\frac{e^{int}}{z^4+1}, e^{i\frac{3\pi}{4}}\right) = \frac{e^{in(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})}}{4 \cdot e^{i\frac{3\pi}{4}}} = \frac{e^{-n\frac{\sqrt{2}}{2} - i\frac{n\sqrt{2}}{2}}}{4 \cdot (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})}$$

$$\begin{aligned} S &= \frac{e^{-n\frac{\sqrt{2}}{2} + i\frac{n\sqrt{2}}{2}}}{4(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})} + \frac{e^{-n\frac{\sqrt{2}}{2} - i\frac{n\sqrt{2}}{2}}}{4(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})} \\ &= \frac{e^{-n\frac{\sqrt{2}}{2}} \left(e^{i\frac{n\sqrt{2}}{2} \cdot (1+i)} + e^{-i\frac{n\sqrt{2}}{2} \cdot (i-1)} \right)}{2\sqrt{2} \cdot (1+i)(i-1)} \\ &= \frac{e^{-n\frac{\sqrt{2}}{2}} \left((1+i)e^{i\frac{n\sqrt{2}}{2}} + (i-1)e^{-i\frac{n\sqrt{2}}{2}} \right)}{2\sqrt{2} \cdot (-2)} \end{aligned}$$

$$\frac{9\pi}{4} = 2\pi + \frac{\pi}{4}$$

$$\int_{-\infty}^{\infty} \frac{\cos nt + i \sin nt}{t^4+1} dt = 2\pi i \cdot \frac{e^{-n\frac{\sqrt{2}}{2}} \left((1+i)e^{i\frac{n\sqrt{2}}{2}} + (i-1)e^{-i\frac{n\sqrt{2}}{2}} \right)}{-4\sqrt{2}}$$

значение
помимо

$$2 \cdot \int_0^{\infty} \frac{\cos nt}{t^4+1} dt = \frac{\sqrt{1}}{2\sqrt{2}} \cdot e^{-n\frac{\sqrt{2}}{2}} \underbrace{\text{Re} \left((1-i)e^{i\frac{n\sqrt{2}}{2}} + (1+i)e^{-i\frac{n\sqrt{2}}{2}} \right)}_{\begin{aligned} &(\cos \frac{n\sqrt{2}}{2} + i \sin \frac{n\sqrt{2}}{2})(1-i) \\ &+ (\cos \frac{n\sqrt{2}}{2} - i \sin \frac{n\sqrt{2}}{2})(1+i) \end{aligned}}$$

$$\int_0^{+\infty} \frac{\cos nt}{t^4+1} dt = \frac{\sqrt{1}}{4\sqrt{2}} e^{-n\frac{\sqrt{2}}{2}} \cdot \left(\cos \frac{n\sqrt{2}}{2} + i \sin \frac{n\sqrt{2}}{2} + \cos \frac{n\sqrt{2}}{2} - i \sin \frac{n\sqrt{2}}{2} \right)$$

$$= \frac{\sqrt{1}}{2\sqrt{2}} e^{-n\frac{\sqrt{2}}{2}} \cdot \left(\cos \frac{n\sqrt{2}}{2} + i \sin \frac{n\sqrt{2}}{2} \right)$$