Uvod u numeričku matematiku

FORMULE

Interpolacija

- Interpolacioni polinom Lagranža

$$L_n(x) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{x - x_j}{x_i - x_j} f(x_i)$$

Drugi oblik: $L_n(x) = \sum_{i=0}^{n} \frac{\omega_{n+1}(x) f(x_i)}{(x - x_i)\omega'_{n+1}(x_i)}, \quad \omega_{n+1}(x) = (x - x_0)(x - x_1)...(x - x_n)$

greška: $|R_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |\omega_{n+1}(x)|, \quad M_{n+1} = \max_{x \in [x_0, x_n]} |f^{(n+1)}(x)|$

- Njutnov interpolacioni polinom sa podeljenim razlikama

podeljene razlike reda 0: $f[x_{i_0}] = f(x_{i_0})$

podeljene razlike reda 1: $f[x_{i_0}, x_{i_1}] = \frac{f(x_{i_1}) - f(x_{i_0})}{x_{i_1} - x_{i_0}}$

podeljene razlike reda k: $f[x_{i_0}, ..., x_{i_k}] = \frac{f[x_{i_1}, ..., x_{i_k}] - f[x_{i_0}, ..., x_{i_{k-1}}]}{x_{i_k} - x_{i_0}}$

$L_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + ... + f[x_0, x_1, ..., x_n](x - x_0)(x - x_1)...(x - x_{n-1})$

greška: $|R_n(x)| \leq |f[x_0, ..., x_n, x]\omega_{n+1}(x)|$

- I Njutnov interpolacioni polinom sa konačnim razlikama

konačne razlike reda 1: $\Delta f_i = f_{i+1} - f_i$

konačne razlike reda k: $\Delta^k f_i = \Delta(\Delta^{k-1} f_i) = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i$

$P_n^I(x) = f_0 + q\Delta f_0 + \frac{q(q-1)}{2!}\Delta^2 f_0 + ... + \frac{q(q-1)...(q-n+1)}{n!}\Delta^n f_0, \quad q = \frac{x - x_0}{h}$

greška: $|R_n(x)| \leq \frac{|\Delta^{n+1} f|}{(n+1)!} |q(q-1)...(q-n)|,$

$\Delta^{n+1} f = \max_i |\Delta^{n+1} f_i| + 2^{n+1}\varepsilon$, ε − greška računa vrednosti funkcije f

- II Njutnov interpolacioni polinom sa konačnim razlikama

$P_n^{II}(x) = f_n + q\Delta f_{n-1} + \frac{q(q+1)}{2!}\Delta^2 f_{n-2} + ... + \frac{q(q+1)...(q+n-1)}{n!}\Delta^n f_0, \quad q = \frac{x - x_n}{h}$

greška: $|R_n(x)| \leq \frac{|\Delta^{n+1} f|}{(n+1)!} |q(q+1)...(q+n)|$

Numeričko diferenciranje

- Prvi i drugi izvod I Njutnovog interpolacionog polinoma sa konačnim razlikama

$$\left(P_n^I(x)\right)' = \frac{1}{h}\left(\Delta f_0 + \frac{2q-1}{2}\Delta^2 f_0 + \frac{3q^2-6q+2}{6}\Delta^3 f_0 + \frac{4q^3-18q^2+22q-6}{24}\Delta^4 f_0 + ...\right), \quad q = \frac{x-x_0}{h}$$

$$\left(P_n^I(x)\right)'' = \frac{1}{h^2}\left(\Delta^2 f_0 + (q-1)\Delta^3 f_0 + \frac{12q^2-36q+22}{24}\Delta^4 f_0 + ...\right), \quad q = \frac{x-x_0}{h}$$

- Prvi i drugi izvod II Njutnovog interpolacionog polinoma sa konačnim razlikama

$$\left(P_n^{II}(x)\right)' = \frac{1}{h}\left(\Delta f_{n-1} + \frac{2q+1}{2}\Delta^2 f_{n-2} + \frac{3q^2+6q+2}{6}\Delta^3 f_{n-3} + \frac{4q^3+18q^2+22q+6}{24}\Delta^4 f_{n-4} + ...\right), \; q = \frac{x-x_n}{h}$$

$$\left(P_n^{II}(x)\right)'' = \frac{1}{h^2}\left(\Delta^2 f_{n-2} + (q+1)\Delta^3 f_{n-3} + \frac{12q^2+36q+22}{24}\Delta^4 f_{n-4} + ...\right), \quad q = \frac{x-x_n}{h}$$

Numerička integracija

- Opšta formula pravougaonika

  h - dužina podintervala

  $$S_0^h(f) = h\sum_{i=0}^{n-1} f\Big(\frac{x_i + x_{i+1}}{2}\Big)$$

  greška: $|R_0^h(f)| \leq \dfrac{b-a}{24}h^2 M_2$

- Opšta formula trapeza

  h - dužina podintervala

  $$S_1^h(f) = \frac{h}{2}\big[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\big]$$

  greška: $|R_1^h(f)| \leq \dfrac{b-a}{12}h^2 M_2$

- Opšta formula Simpsona

  h - dužina podintervala

  $$S_2^h(f) = \frac{h}{3}\big[f(x_0) + 4\sum_{i=1}^{n} f(x_{2i-1}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + f(x_{2n})\big]$$

  greška: $|R_2^h(f)| \leq \dfrac{b-a}{180}h^4 M_4$

- Rungeova ocena greške

  $$|I(f) - S^{\frac{h}{2}}(f)| \approx \frac{|S^h(f) - S^{\frac{h}{2}}(f)|}{2^k - 1}$$

  Simpsonova kvadraturna formula: $k = 4$

  Formula pravougaonika i trapezna kvadraturna formula: $k = 2$

- Rekurentna formula za formiranje sistema ortogonalnih polinoma

  $Q_{-1}(x) = 0,$
  $Q_0(x) = 1,$
  $Q_{k+1}(x) = \Big(x - \dfrac{(xQ_k, Q_k)}{(Q_k, Q_k)}\Big)Q_k(x) - \dfrac{(Q_k, Q_k)}{(Q_{k-1}, Q_{k-1})}Q_{k-1}(x), \quad k = 0, 1, \ldots$

- Ležandrovi polinomi

  eksplicitna formula: $L_n(x) = \dfrac{1}{2^n\ n!}\dfrac{d^n}{dx^n}\big((x^2-1)^n\big),\ n = 0, 1\ldots$
  rekurentna formula:
  $L_0(x) = 1,$
  $L_1(x) = x,$
  $L_n(x) = \dfrac{2n-1}{n}xL_{n-1}(x) - \dfrac{n-1}{n}L_{n-2}(x), \quad n = 2, 3, \ldots$

- Čebiševljevi polinomi

  eksplicitna formula: $T_n(x) = \cos(n \arccos x),\ n = 0, 1\ldots$
  rekurentna formula:
  $T_0(x) = 1,$
  $T_1(x) = x,$
  $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots$

Nelinearne jednačine

- Metoda regula-falsi (lažnog položaja)

  $x_{n+1} = x_n - \dfrac{f(x_n)}{f(x_F) - f(x_n)}(x_F - x_n)$
  Kriterijum zaustavljanja: $|x_n - x_{n-1}| \leq \dfrac{m_1\varepsilon}{M_1 - m_1}$

- Metoda sečice

  $x_{n+1} = x_n - \dfrac{f(x_n)}{f(x_{n-1}) - f(x_n)}(x_{n-1} - x_n)$
  Kriterijum zaustavljanja: $|x_n - x_{n-1}| \leq \dfrac{m_1\varepsilon}{M_1 - m_1}$

- Njutnova metoda tangente

  $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
  Kriterijum zaustavljanja: $|x_n - x_{n-1}| \leq \sqrt{\dfrac{2m_1\varepsilon}{M_2}}$