

ЗАДАТАК 0.1. Израчунати $\int_0^1 \frac{dx}{\sqrt{2x^2+4x+3}}$.

Решење.

$$\begin{aligned} I &= \int_0^1 \frac{dx}{\sqrt{2x^2+4x+3}} = \frac{1}{\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{x^2+2x+\frac{3}{2}}} = \frac{1}{\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{(x+1)^2+\frac{1}{2}}} \\ &= \left(\begin{array}{l} x+1=t \\ dx=dt \end{array} \right) = \frac{1}{\sqrt{2}} \int_1^2 \frac{dt}{\sqrt{t^2+\frac{1}{2}}} = \frac{1}{\sqrt{2}} \ln \left(t + \sqrt{t^2+\frac{1}{2}} \right) \Big|_1^2 \\ &= \frac{1}{\sqrt{2}} \ln \frac{2+\sqrt{\frac{9}{2}}}{1+\sqrt{\frac{3}{2}}} = \frac{1}{\sqrt{2}} \ln \frac{2\sqrt{2}+3}{\sqrt{2}+\sqrt{3}} = \frac{1}{\sqrt{2}} \ln (2\sqrt{6}-4+3\sqrt{3}-3\sqrt{2}). \end{aligned}$$

△

ЗАДАТАК 0.2. Израчунати $I = \int_2^3 \frac{dx}{x^3+x^2-2}$.

Решење.

$$\int_2^3 \frac{dx}{x^3+x^2-2} = \int_2^3 \frac{dx}{(x-1)(x^2+2x+2)},$$

Множењем једнакости

$$\frac{1}{x^3+x^2-2} = \frac{1}{(x-1)(x^2+2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}$$

са $(x+1)(x^2+2x+2)$ добијамо систем

$$A+B=0, \quad 2A-B+C=0, \quad 2A-C=1$$

чије је решење $A = \frac{1}{5}$, $B = -\frac{1}{5}$, $C = -\frac{3}{5}$. Према томе, интеграл је једнак

$$\int_2^3 \left(\frac{\frac{1}{5}}{x-1} + \frac{-\frac{1}{5}x - \frac{3}{5}}{x^2+2x+2} \right) dx = \frac{1}{5} \int_2^3 \frac{dx}{x-1} - \frac{1}{5} \int_2^3 \frac{x+3}{x^2+2x+2} dx = \frac{1}{5} I_1 - \frac{1}{5} I_2,$$

$$I_1 = \int_2^3 \frac{dx}{x-1} = \ln(x-1) \Big|_2^3 = \ln 2,$$

$$\begin{aligned} I_2 &= \int_2^3 \frac{x+3}{x^2+2x+2} dx = \frac{1}{2} \int_2^3 \frac{2x+6}{x^2+2x+2} dx = \frac{1}{2} \int_2^3 \frac{2x+2}{x^2+2x+2} dx + \frac{1}{2} \int_2^3 \frac{4}{x^2+2x+2} dx \\ &= \frac{1}{2} \ln(x^2+2x+2) \Big|_2^3 + 2 \int_2^3 \frac{dx}{(x+1)^2+1} = \frac{1}{2} \ln \frac{17}{10} + 2 \operatorname{arctg}(x+1) \Big|_2^3 \\ &= \frac{1}{2} \ln \frac{17}{10} + 2 \operatorname{arctg} 4 - 2 \operatorname{arctg} 3. \end{aligned}$$

Добили смо

$$I = \frac{1}{5}I_1 - \frac{1}{5}I_2 = \frac{1}{5} \ln 2 - \frac{1}{10} \ln \frac{17}{10} - \frac{2}{5} \operatorname{arctg} 4 + \frac{2}{5} \operatorname{arctg} 3.$$

△

ЗАДАТАК 0.3. Израчунати $I = \int x^2 \operatorname{arctg} \frac{1}{x} dx$.

Решење.

$$\begin{aligned} I &= \int x^2 \operatorname{arctg} \frac{1}{x} dx = \left(\begin{array}{l} u = \operatorname{arctg} \frac{1}{x} \Rightarrow du = \frac{1}{1+\frac{1}{x^2}} \left(-\frac{1}{x^2}\right) dx = -\frac{dx}{x^2+1} \\ dv = x^2 dx \Rightarrow v = \frac{x^3}{3} \end{array} \right) \\ &= \frac{x^3}{3} \cdot \operatorname{arctg} \frac{1}{x} + \int \frac{x^3}{3} \frac{dx}{x^2+1} = \frac{x^3}{3} \cdot \operatorname{arctg} \frac{1}{x} + \frac{1}{3} \int \frac{x^3 dx}{x^2+1} = \frac{x^3}{3} \cdot \operatorname{arctg} \frac{1}{x} + \frac{1}{3} I_1, \end{aligned}$$

$$\begin{aligned} I_1 &= \int \frac{x^3 dx}{x^2+1} = \left(\begin{array}{l} x^2+1=t \\ 2x dx = dt \end{array} \right) = \frac{1}{2} \int \frac{(t-1)dt}{t} = \frac{1}{2} (t - \ln |t|) + C \\ &= \frac{1}{2} (x^2+1 - \ln |x^2+1|) + C, \end{aligned}$$

$$I = \frac{1}{3} x^3 \operatorname{arctg} \frac{1}{x} + \frac{1}{6} (x^2+1) - \frac{1}{6} \ln(x^2+1) + C.$$

△

ЗАДАТАК 0.4. Израчунати $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{13 + \operatorname{arctg} x + e^x \cos^2 x}{\cos x} dx$.

Решење. Важи

$$\begin{aligned} I &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{13 + \operatorname{arctg} x + e^x \cos^2 x}{\cos x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{13}{\cos x} + \frac{\operatorname{arctg} x}{\cos x} + e^x \cos x \right) dx \\ &= 13 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\cos x} + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\operatorname{arctg} x}{\cos x} dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^x \cos x dx = I_1 + I_2 + I_3, \end{aligned}$$

при чему је

$$\begin{aligned} I_1 &= 13 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\cos x} = 13 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x dx}{\cos^2 x} = \left(\begin{array}{l} \sin x = t \\ \cos x dx = dt \end{array} \right) = 13 \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{dt}{1-t^2} = \frac{13}{2} \ln \frac{1+x}{1-x} \Big|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \\ &= \frac{13}{2} \left(\ln \frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} - \ln \frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} \right) = \frac{13}{2} \left(\ln \frac{2+\sqrt{2}}{2-\sqrt{2}} - \ln \frac{2-\sqrt{2}}{2+\sqrt{2}} \right) = 13 \ln \frac{2+\sqrt{2}}{2-\sqrt{2}}, \end{aligned}$$

$$I_2 = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\operatorname{arctg} x}{\cos x} dx = 0.$$

Урадимо неодређени интеграл

$$\begin{aligned} I_3 &= \int e^x \cos x dx = \left(\begin{array}{l} u = \cos x \Rightarrow du = -\sin x dx \\ dv = e^x \Rightarrow v = e^x \end{array} \right) \\ &= e^x \cos x + \int e^x \sin x dx = \left(\begin{array}{l} u = \sin x \Rightarrow du = \cos x \\ dv = e^x \Rightarrow v = e^x \end{array} \right) \\ &= e^x \cos x + \left(e^x \sin x - \int \cos x e^x \right) \\ &= e^x \cos x + e^x \sin x - \int \cos x e^x = e^x \cos x + e^x \sin x - I_3. \end{aligned}$$

Добили смо да је

$$I_3 = \frac{1}{2} e^x (\cos x + \sin x) + C,$$

одакле је

$$I_3 = \frac{1}{2} e^x (\cos x + \sin x) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{1}{2} e^{\frac{\pi}{4}} \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}.$$

Према томе,

$$I = 13 \ln \frac{2 + \sqrt{2}}{2 - \sqrt{2}} + \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}.$$

△

ЗАДАТАК 0.5. Израчунати обим и површину фигуре ограничене кривом $(x-1)^2 + y^2 = 4$ и правама $y = 0$, $x = 1$ и $\sqrt{3}x + 3y - 3\sqrt{3} = 0$. Затим, одредити запремину тела које се добија ротацијом дате фигуре око x -осе.

Решење.

$$\begin{aligned} P &= \int_{-1}^0 \sqrt{4 - (x-1)^2} dx + \frac{\frac{2\sqrt{3}}{3} + \sqrt{3}}{2} = \int_{-1}^0 \sqrt{4 - (x-1)^2} dx + \frac{5\sqrt{3}}{6} = \left(\begin{array}{l} x-1 = t \\ dx = dt \end{array} \right) \\ &= \int_{-2}^{-1} \sqrt{4 - t^2} dt + \frac{5\sqrt{3}}{6} = \left(\begin{array}{l} t = 2 \sin u \\ dt = 2 \cos u du \end{array} \right) = 4 \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} \cos^2 u du + \frac{5\sqrt{3}}{6} \\ &= 2 \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} (1 + \cos 2u) du + \frac{5\sqrt{3}}{6} = 2 \left(u + \frac{1}{2} \sin 2u \right) \Big|_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} + \frac{5\sqrt{3}}{6} = (2u + \sin 2u) \Big|_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} + \frac{5\sqrt{3}}{6} \\ &= 2 \left(-\frac{\pi}{6} + \frac{\pi}{2} \right) - \frac{1}{2} + \frac{5\sqrt{3}}{6} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2} + \frac{5\sqrt{3}}{6} = \frac{2\pi}{3} + \frac{\sqrt{3}}{3}. \end{aligned}$$

$$\begin{aligned} O &= O_1 + O_2 + O_3 + O_4 = 2 + \frac{2\sqrt{3}}{3} + \sqrt{(0-1)^2 + \left(\sqrt{3} - \frac{2\sqrt{3}}{3} \right)^2} + O_4 \\ &= 2 + \frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{3} + O_4 = 2 + \frac{4\sqrt{3}}{3} + O_4, \end{aligned}$$

$$\begin{aligned}
O_4 &= \int_{-1}^0 \sqrt{1 + [y'(x)]^2} dx = \int_{-1}^0 \sqrt{1 + \left[\left(\sqrt{4 - (x-1)^2} \right)' \right]^2} dx \\
&= \int_{-1}^0 \sqrt{1 + \left[\frac{-2(x-1)}{2\sqrt{4 - (x-1)^2}} \right]^2} dx = \int_{-1}^0 \sqrt{1 + \frac{(x^2 - 2x + 1)}{4 - (x-1)^2}} dx \\
&= \int_{-1}^0 \sqrt{\frac{4 - x^2 + 2x - 1 + x^2 - 2x + 1}{4 - (x-1)^2}} dx = 2 \int_{-1}^0 \frac{1}{\sqrt{4 - (x-1)^2}} dx \quad \left(\begin{array}{l} t = x - 1 \\ dt = dx \end{array} \right) \\
&= 2 \int_{-2}^{-1} \frac{1}{\sqrt{4 - t^2}} dt = 2 \arcsin \frac{t}{2} \Big|_{-2}^{-1} = 2 \left(-\frac{\pi}{6} + \frac{\pi}{2} \right) = \frac{2\pi}{3},
\end{aligned}$$

$$O = 2 + \frac{4\sqrt{3}}{3} + O_4 = 2 + \frac{4\sqrt{3}}{3} + \frac{2\pi}{3}.$$

$$\begin{aligned}
V &= V_1 + V_2 = \pi \int_{-1}^0 \left(\sqrt{4 - (x-1)^2} \right)^2 dx + \pi \int_0^1 \left(\sqrt{3} - \frac{\sqrt{3}}{3}x \right)^2 dx \\
&= \pi \int_{-1}^0 (3 + 2x - x^2) dx + \pi \int_0^1 \left(3 - 2x + \frac{1}{3}x^2 \right) dx \\
&= \pi \left(3x + x^2 - \frac{x^3}{3} \right) \Big|_{-1}^0 + \pi \left(3x - x^2 - \frac{x^3}{9} \right) \Big|_0^1 = \pi \left(3 - 1 - \frac{1}{3} \right) + \pi \left(3 - 1 - \frac{1}{9} \right) \\
&= \frac{34}{9} \pi.
\end{aligned}$$

△