1. $$f(x)=\frac{x^2+4x+4}{|x-5|}e^{\frac{1}{x+2}}=\frac{(x+2)^2}{|x-5|}e^{\frac{1}{x+2}}$$

а) кога $x\to+\infty$ биде $x-5>0$, па имаме

$$f(x)=\frac{(x+2)^2}{x-5}e^{\frac{1}{x+2}}=\frac{x^2+4x+4}{x\left(1-\frac{5}{x}\right)}e^{\frac{1}{x\left(1+\frac{2}{x}\right)}}=$$

$$=\frac{x^2+4x+4}{x}\cdot\left(1-\frac{5}{x}\right)^{-1}\cdot e^{\frac{1}{x}\left(1+\underset{\to 0}{\frac{2}{x}}\right)^{-1}}=$$

$$=\left(x+4+\frac{4}{x}\right)\cdot\left(1+\underset{\to 0}{\frac{-5}{x}}\right)^{-1}\cdot e^{\frac{1}{x}\left(1-\frac{2}{x}+o\left(\frac{1}{x}\right)\right)}=$$

$$=\underbrace{\left(x+4+\frac{4}{x}\right)\left(1+\frac{5}{x}+\frac{25}{x^2}+o\left(\frac{1}{x^2}\right)\right)}\cdot e^{\overset{\to 0}{\frac{1}{x}-\frac{2}{x^2}}+o\left(\frac{1}{x^2}\right)}=$$

$$=\left(x+5+\frac{25}{x}+4+\frac{20}{x}+\frac{4}{x}+o\left(\frac{1}{x^2}\right)\right)\left(1+\frac{1}{x}-\frac{2}{x^2}+\frac{1}{2}\underbrace{\left(\frac{1}{x}-\frac{2}{x^2}\right)^2}_{\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)}+o\left(\frac{1}{x^2}\right)\right)$$

$$=\left(x+9+\frac{49}{x}+o\left(\frac{1}{x}\right)\right)\cdot\left(1+\frac{1}{x}-\frac{2}{x^2}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right)=$$

$$=\left(x+9+\frac{49}{x}+o\left(\frac{1}{x^2}\right)\right)\cdot\left(1+\frac{1}{x}-\frac{3}{2x^2}+o\left(\frac{1}{x^2}\right)\right)=$$

$$= \left(x + 5 + \frac{25}{x} + 4 + \frac{20}{x} + \frac{4}{x} + o\left(\frac{1}{x^2}\right)\right)\left(1 + \frac{1}{x} - \frac{2}{x^2} + \frac{1}{2}\underbrace{\left(\frac{1}{x} - \frac{2}{x^2}\right)^2}_{\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)} + o\left(\frac{1}{x^2}\right)\right)$$

$$= \left(x + 9 + \frac{49}{x} + o\left(\frac{1}{x}\right)\right) \cdot \left(1 + \frac{1}{x} - \frac{2}{x^2} + \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right) =$$

$$= \left(x + 9 + \frac{49}{x} + o\left(\frac{1}{x^2}\right)\right) \cdot \left(1 + \frac{1}{x} - \frac{3}{2x^2} + o\left(\frac{1}{x^2}\right)\right) =$$

$$= x + 1 - \frac{3}{2x} + 9 + \frac{9}{x} + \frac{49}{x} + o\left(\frac{1}{x}\right) = x + 10 + \frac{113}{2x} + o\left(\frac{1}{x}\right), \quad x \to +\infty$$

$$\Rightarrow a_1 = 1, \; b_1 = 10, \; c_1 = \frac{113}{2}$$

Кад $x \to -\infty$ будући $x - 5 < 0$, па имамо

$$f(x) = \frac{x^2 + 4x + 4}{-(x-5)} e^{\frac{1}{x+2}} = -\frac{x^2 + 4x + 4}{x-5} e^{\frac{1}{x+2}} \overset{\text{као за } +\infty}{=}$$

$$= -\left(x + 10 + \frac{116}{2x} + o\left(\frac{1}{x}\right)\right) = -x - 10 - \frac{113}{2x} + o\left(\frac{1}{x}\right), \quad x \to -\infty$$

$$\Rightarrow a_2 = -1, \; b_2 = -10, \; c_2 = -\frac{113}{2}$$

б) 1) домен

$x+2 \neq 0$ и $x-5 \neq 0 \iff x \neq -2$ и $x \neq 5$

$D_f = \mathbb{R} \setminus \{-2, 5\} = (-\infty, -2) \cup (-2, 5) \cup (5, +\infty)$

2) нуле и знак

$$f(x) = 0 \iff \frac{(x+2)^2}{|x-5|} \cdot e^{\frac{1}{x+2}} = 0 \iff$$

$$\iff x+2 = 0 \quad \vee \quad \underbrace{e^{\frac{1}{x+2}}}_{>0 \ \forall x \in D_f} = 0$$

$\iff x = -2 \notin D_f$

$\Rightarrow$ f нема нула

за $x \in D_f$ је $x \neq -2$ и $x \neq 5$, па је

$(x+2)^2 > 0$, $|x-5| > 0$ и $e^{\frac{1}{x+2}} > 0$

$\Rightarrow f(x) > 0 \quad \forall x \in D_f$

$\Rightarrow f(x) > 0 \quad \forall x \in D_f$

3) због домена ф-ја није ни парна, ни непарна, ни периодична (домен није симетричан и не припада му само коначно много тачака)

Надаље је
$$f(x) = \begin{cases} \dfrac{(x+2)^2}{x-5} e^{\frac{1}{x+2}}, & x > 5 \\ -\dfrac{(x+2)^2}{x-5} e^{\frac{1}{x+2}}, & x \in (-\infty,-2) \cup (-2,5) \end{cases}$$

јер је
$$|x-5| = \begin{cases} x-5, & x > 5 \\ -(x-5), & x \in (-\infty,-2) \cup (-2,5) \end{cases}$$

4) асимптоте

В.А. $\lim\limits_{x\to-2-} F(x) = \lim\limits_{x\to-2-} -\frac{\overset{\to 0+}{(x+2)^2}}{\underset{\downarrow\, -7}{(x-5)}}\, e^{\overset{\to -\infty}{\frac{1}{x+2}}} = -\frac{0+}{-7}\cdot \underbrace{e^{-\infty}}_{=0} = 0$

$\lim\limits_{x\to-2+} F(x) = \lim\limits_{x\to-2+} -\frac{\overset{\to 0+}{(x+2)^2}}{\underset{\downarrow\, -7}{(x-5)}}\, e^{\overset{\to +\infty}{\frac{1}{x+2}}} = \frac{1}{7}\lim\limits_{x\to-2+} \frac{e^{\frac{1}{x+2}}}{\frac{1}{(x+2)^2}} \overset{\frac{\infty}{\infty}\ \text{Л.П.}}{=}$

$= \frac{1}{7}\lim\limits_{x\to-2+} \frac{e^{\frac{1}{x+2}}\cdot\frac{-1}{(x+2)^2}}{-\frac{2}{(x+2)^3}} = \frac{1}{14}\lim\limits_{x\to-2+} \frac{e^{\frac{1}{x+2}}}{\frac{1}{x+2}} \overset{\frac{\infty}{\infty}\ \text{Л.П.}}{=}$

$= \frac{1}{14}\lim\limits_{x\to-2+} \frac{\overset{\to+\infty}{e^{\frac{1}{x+2}}}\cdot\frac{-\cancel{1}}{\cancel{(x+2)^2}}}{-\frac{\cancel{1}}{\cancel{(x+2)^2}}} = +\infty$ $\Rightarrow$ права $x=-2$ јесте В.А. фје F, $x\to-2+$,
а кад $x\to-2-$ фја се приближава тачки $(-2,0)$

За $x\to+\infty$ из дела а) имамо да је $y=x+10$ К.А.
и за $x\to-\infty$ је права $y=-x-10$ К.А.

а кад $x \to -2^-$ ф-ја се приближава тачки $(-2,0)$

за $x \to +\infty$ из дела а) имамо да је $y = x+10$ К.А.

и за $x \to -\infty$ је права $y = -x-10$ К.А.

5) монотоност и лок. екстремуми

$x \in (5,+\infty)$:

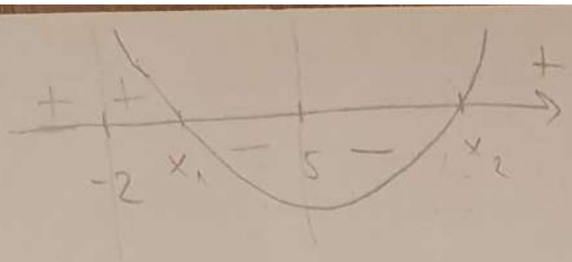
$$f'(x) = \frac{2(x+2)\cdot(x-5) - (x+2)^2\cdot 1}{(x-5)^2} e^{\frac{1}{x+2}} +$$

$$+ \frac{\cancel{(x+2)^2}}{x-5} \cdot e^{\frac{1}{x+2}} \cdot \frac{-1}{\cancel{(x+2)^2}} = \frac{2x^2-6x-20-(x^2+4x+4)}{(x-5)^2} e^{\frac{1}{x+2}} -$$

$$- \frac{1}{x-5} e^{\frac{1}{x+2}} \;\; \Big| \cdot (x-5) = \frac{x^2-10x-24-(x-5)}{(x-5)^2} e^{\frac{1}{x+2}} = \frac{x^2-11x-19}{\underbrace{(x-5)^2}_{>0}} \underbrace{e^{\frac{1}{x+2}}}_{>0}$$

$$f'(x) > 0 \iff x^2 - 11x - 19 > 0$$

$$x^2 - 11x - 19 = 0 \iff x_{1,2} = \frac{11 \pm \overbrace{\sqrt{197}}^{\approx 14}}{2} \qquad x_1 \approx -\frac{3}{2}, \; x_2 \approx \frac{25}{2}$$



$\Rightarrow f'(x) > 0$ за $x \in (x_2, +\infty)$

и $f'(x) < 0$ за $x \in (5, x_2)$

$\Rightarrow f \nearrow$ на $\left(\frac{11+\sqrt{197}}{2}, +\infty\right)$ и $f \searrow$ на $\left(5, \frac{11+\sqrt{197}}{2}\right)$

па је тачка $x_2 = \frac{11+\sqrt{197}}{2}$ лок. минимум

$x \in (-\infty, -2) \cup (-2, 5)$: $\quad f'(x) = \left(-\frac{(x+2)^2}{x-5} e^{\frac{1}{x+2}}\right)' =$

користимо рачун од малопре $\leftarrow$ $= -\frac{x^2-11x-19}{\underbrace{(x-5)^2}_{>0}} \underbrace{e^{\frac{1}{x+2}}}_{>0}$

$f'(x) > 0 \Leftrightarrow x^2 - 11x - 19 < 0$

$>0 \qquad >0$

$f'(x) > 0 \iff x^2 - 11x - 19 < 0$

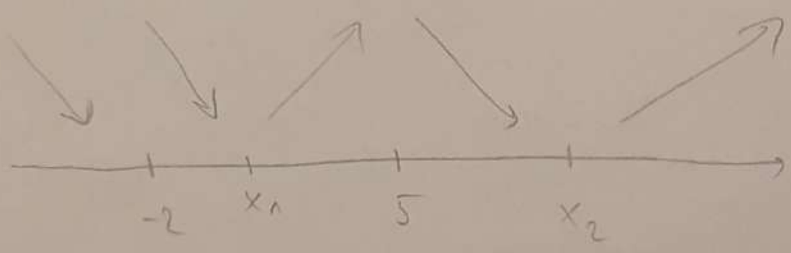
$\Rightarrow f'(x) < 0$ за $x \in (-\infty, -2) \cup (-2, x_1)$

и $f'(x) > 0$ за $x \in (x_1, 5)$

$\Rightarrow f \searrow$ на $(-\infty, -2)$ и $f \searrow$ на $\left(-2, \frac{11-\sqrt{197}}{2}\right)$

и $f \nearrow$ на $\left(\frac{11-\sqrt{197}}{2}, 5\right)$, т.е. точка

$x_1 = \frac{11-\sqrt{197}}{2}$ лок. минимум



б) конвексност, конкавност и превојне тачке

$x \in (5, +\infty)$:

$$f''(x) = \frac{(2x-11)(x-5)^2 - (x^2-11x-19)\cdot 2(x-5)}{(x-5)^4} e^{\frac{1}{x+2}} + \quad |:(x-5)$$

$$+ \frac{x^2-11x-19}{(x-5)^2} \cdot e^{\frac{1}{x+2}} \cdot \frac{-1}{(x+2)^2} = \frac{\cancel{2x^2} - 21x + 55 - \cancel{2x^2} + 22x + 38}{(x-5)^3} e^{\frac{1}{x+2}}$$

$$- \frac{x^2-11x-19}{(x-5)^2(x+2)^2} e^{\frac{1}{x+2}} = \frac{x+93}{(x-5)^3} e^{\frac{1}{x+2}} \;|\cdot(x+2)^2 \; - \frac{x^2-11x-19}{(x-5)^2(x+2)^2} e^{\frac{1}{x+2}} \;|\cdot(x-5) =$$

$$= \frac{(x+93)(x^2+4x+4) - (x^2-11x-19)(x-5)}{(x-5)^3(x+2)^2} e^{\frac{1}{x+2}} =$$

$$= \frac{\cancel{x^3} + 97x^2 + 376x + 372 - (\cancel{x^3} - 16x^2 + 36x + 95)}{(x-5)^3(x+2)^2} e^{\frac{1}{x+2}} =$$

$$= \frac{\overbrace{113x^2 + 340x + 277}^{>0}}{\underbrace{(x-5)^3}_{>0}\underbrace{(x+2)^2}_{>0}} \underbrace{e^{\frac{1}{x+2}}}_{>0} > 0 \Rightarrow f \text{ је конвексна на } (5, +\infty)$$

$\ldots 4 \cdot 113 \cdot 277 < 0$

$$= \frac{x^3+97x^2+376x+372-(x^3-16x^2+36x+95)}{(x-5)^3(x+2)^2}\, e^{\frac{1}{x+2}} =$$

$$= \frac{\overbrace{113x^2+340x+277}^{>0}}{\underbrace{(x-5)^3}_{>0}\underbrace{(x+2)^2}_{>0}}\, \underbrace{e^{\frac{1}{x+2}}}_{>0} > 0 \Rightarrow f \text{ је конвексна на } (5,+\infty)$$

$$D = 340^2 - 4\cdot 113\cdot 277 < 0$$

$\underline{x\in(-\infty,-2)\cup(-2,5):}$

$$f''(x) = -\frac{\overbrace{113x^2+340x+277}^{>0}}{\underbrace{(x-5)^3}_{<0}\underbrace{(x+2)^2}_{>0}}\, \underbrace{e^{\frac{1}{x+2}}}_{>0} > 0$$

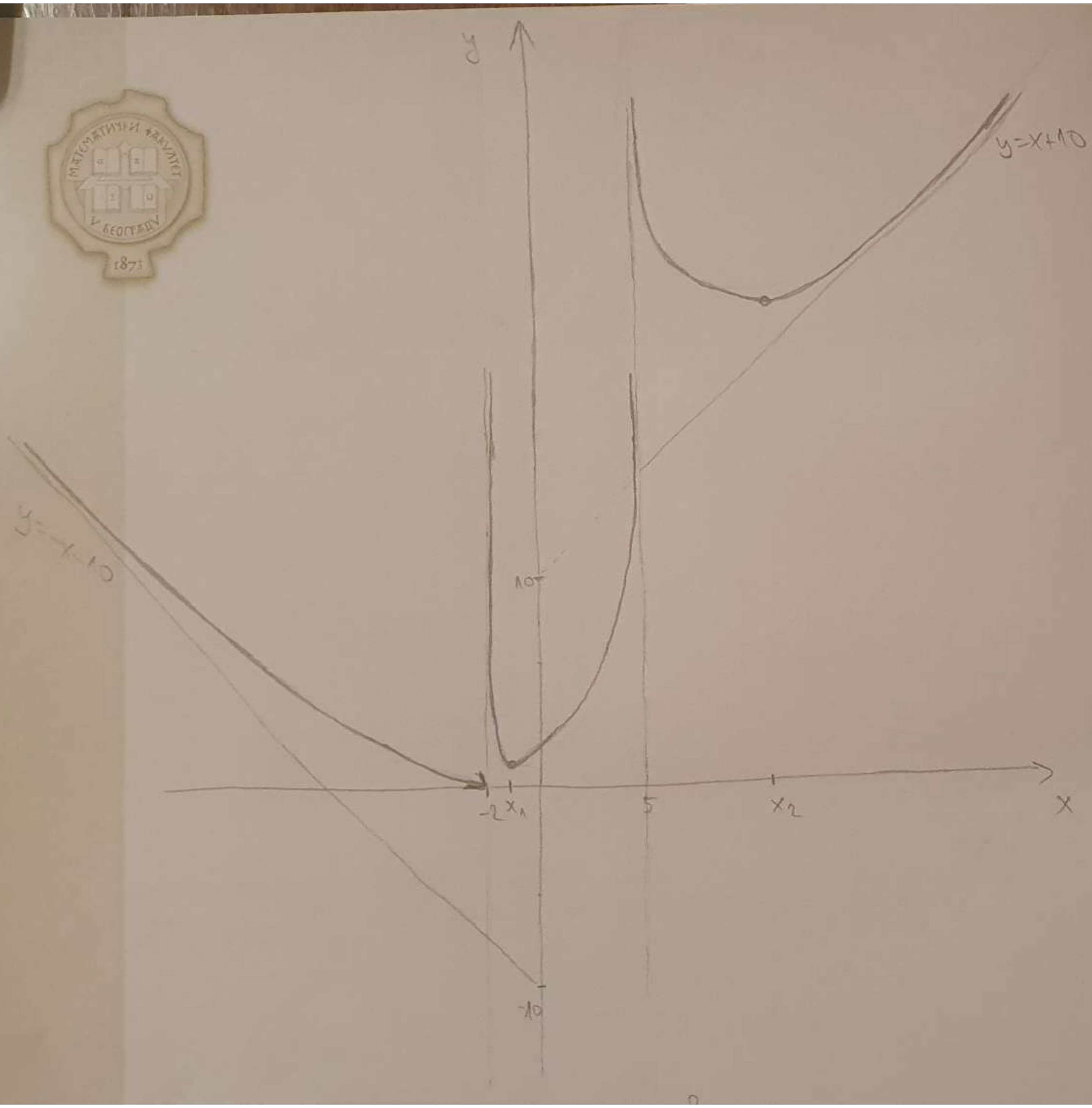
$\Rightarrow f$ је конвексна на $(-\infty,-2)\cup(-2,5)$

7) График

Због $c_1>0$ и $c_2<0$ имамо да се фја f приближава својим косим асимптотама са горње стране

$$\lim_{x\to-2-} f'(x) = \lim_{x\to-2-} -\frac{x^2-11x-19}{(x-5)^2}\, e^{\overbrace{\frac{1}{x+2}}^{\to-\infty}} = -\frac{7}{49}\cdot e^{-\infty} = 0$$

$\Rightarrow f$ „улази" хоризонтално у тачку $(-2,0)$ с леве стране


y
y=x+10
y=-x-10
10
-2
x1
5
x2
x
-10

б) $\alpha \leq 0$: 0 решења

$\alpha \in (0, f(x_1))$: 1 решење

$\alpha = f(x_1)$: 2 решења

$\alpha \in (f(x_1), f(x_2))$: 3 решења

$\alpha = f(x_2)$: 4 решења

$\alpha > f(x_2)$: 5 решења

$$2.\quad I = \int_{-\frac{\pi}{2}}^{7\pi/4} \underbrace{\frac{|\sin 2x|}{\cos^4 x + \sin^4 x}}_{f(x)} dx = \underbrace{\int_{-\pi/2}^{0} f(x)dx}_{I_1} + \underbrace{\int_{0}^{\pi/2} f(x)dx}_{I_2} + \underbrace{\int_{\pi/2}^{\pi} f(x)dx}_{I_3}$$

$$+ \underbrace{\int_{\pi}^{3\pi/2} f(x)dx}_{I_4} + \underbrace{\int_{3\pi/2}^{7\pi/4} f(x)dx}_{I_5}$$

На ове интервале смо делили због знака ф-је $\sin 2x$:

$\sin 2x \le 0$ за $x \in [-\frac{\pi}{2}, 0]$ тј. $2x \in [-\pi, 0]$

$\sin 2x \ge 0$ за $x \in [0, \frac{\pi}{2}]$ тј. $2x \in [0, \pi]$

$\sin 2x \le 0$ за $x \in [\frac{\pi}{2}, \pi]$ тј. $2x \in [\pi, 2\pi]$

$\sin 2x \ge 0$ за $x \in [\pi, \frac{3\pi}{2}]$ тј. $2x \in [2\pi, 3\pi]$

$\sin 2x \le 0$ за $x \in [\frac{3\pi}{2}, \frac{7\pi}{4}]$ тј. $2x \in [3\pi, \frac{7\pi}{2}]$.

$$I_1 = -\int_{-\pi/2}^{0} \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx = -2\int_{-\pi/2}^{0} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx = \left(\begin{array}{ll} t = \sin x & \begin{array}{c|c|c} x & -\frac{\pi}{2} & 0 \\ \hline t & -1 & 0 \end{array} \\ dt = \underbrace{\cos x}_{\ge 0 \text{ на } [-\frac{\pi}{2}, 0]} dx & \end{array} \right)$$

за $x \in [\pi, \frac{3\pi}{2}]$ тј. $2x \in [2\pi, 3\pi]$

$\sin 2x \le 0$ за $x \in [\frac{3\pi}{2}, \frac{7\pi}{4}]$ тј. $2x \in [3\pi, \frac{7\pi}{2}]$.

$$I_1 = -\int_{-\pi/2}^{0} \frac{\sin 2x}{\cos^4 x + \sin^4 x}\,dx = -2\int_{-\pi/2}^{0} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx = \left(\begin{array}{l} t = \sin x \\ dt = \underbrace{\cos x}_{\ge 0 \text{ на } [-\frac{\pi}{2}, 0]} dx \end{array} \quad \begin{array}{c|c|c} x & -\frac{\pi}{2} & 0 \\ \hline t & -1 & 0 \end{array} \right)$$

$$= -2\int_{-1}^{0} \frac{t\,dt}{(1-t^2)^2 + t^4} = -2\int_{-1}^{0} \frac{t\,dt}{2t^4 - 2t^2 + 1} = \left(\begin{array}{l} s = t^2 \\ ds = \underbrace{2t}_{\le 0 \text{ на } [-1,0]} dt \end{array} \quad \begin{array}{c|c|c} t & -1 & 0 \\ \hline s & 1 & 0 \end{array} \right)$$

$$= -\int_{1}^{0} \frac{ds}{2s^2 - 2s + 1} = \frac{1}{2}\int_{0}^{1} \frac{ds}{s^2 - s + \frac{1}{2}} = \frac{1}{2}\int_{0}^{1} \frac{ds}{(s-\frac{1}{2})^2 + \frac{1}{4}} =$$

$$= \left(\begin{array}{l} u = s - \frac{1}{2} \\ du = ds \end{array} \quad \begin{array}{c|c|c} s & 0 & 1 \\ \hline u & -\frac{1}{2} & \frac{1}{2} \end{array} \right) = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{\frac{du}{u^2 + \frac{1}{4}}}_{\text{парна}} = \frac{1}{2}\cdot 2\int_{0}^{1/2} \frac{du}{u^2 + \frac{1}{4}} =$$

$$= \int_{0}^{1/2} \frac{du}{u^2 + (\frac{1}{2})^2} = 2\,\mathrm{arctg}\, 2u \Big|_{0}^{1/2} = 2(\mathrm{arctg}\, 1 - \mathrm{arctg}\, 0) = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$$

Аналогно добијамо $\boxed{I_2 = I_3 = I_4 = \frac{\pi}{2}}$

$$I_5 = -\int_{3\pi/2}^{7\pi/4} \frac{\sin 2x}{\cos^4 x + \sin^4 x} = -2\int_{3\pi/2}^{7\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx =$$

$$= \left(\begin{array}{l} t = \sin x \\ dt = \underbrace{\cos x}_{\geq 0 \text{ на } [\frac{3\pi}{2}, \frac{7\pi}{4}]} dx \end{array} \quad \begin{array}{c|c|c} x & 3\pi/2 & 7\pi/4 \\ \hline t & -1 & -\frac{1}{\sqrt{2}} \end{array} \right) = -2\int_{-1}^{-1/\sqrt{2}} \frac{t\,dt}{(1-t^2)^2 + t^4} =$$

$$= \left(\begin{array}{l} s = t^2 \\ ds = 2\underbrace{t}_{\leq 0 \text{ на } [-1, -\frac{1}{\sqrt{2}}]} dt \end{array} \quad \begin{array}{c|c|c} t & -1 & -\frac{1}{\sqrt{2}} \\ \hline s & 1 & \frac{1}{2} \end{array} \right) = -\int_{1}^{1/2} \frac{ds}{2s^2 - 2s + 1} =$$

$$= \frac{1}{2}\int_{1/2}^{1} \frac{ds}{\left(s - \frac{1}{2}\right)^2 + \frac{1}{4}} = \left(u = s - \frac{1}{2} \right) = \frac{1}{2}\int_{0}^{1/2} \frac{du}{u^2 + \left(\frac{1}{2}\right)^2} =$$

$$= \frac{1}{2} \cdot 2 \operatorname{arctg} 2u \Big|_{0}^{1/2} = \operatorname{arctg} 1 - \operatorname{arctg} 0 = \frac{\pi}{4}$$

$$= \frac{1}{2} \int_{1/2} \frac{ds}{\left(s-\frac{1}{2}\right)^2+\frac{1}{4}} = \left(u = s - \frac{1}{2} \right) = \frac{1}{2} \int_0^{1/2} \frac{du}{u^2+\left(\frac{1}{2}\right)^2} =$$

$$= \frac{1}{2} \cdot 2 \operatorname{arctg} 2u \Big|_0^{1/2} = \operatorname{arctg} 1 - \operatorname{arctg} 0 = \frac{\pi}{4}$$

$$\Rightarrow I = 4 \cdot \frac{\pi}{2} + \frac{\pi}{4} = \frac{9\pi}{4}$$

<u>2. начин:</u> Уочимо да је $f\left(x+\frac{\pi}{2}\right) = f(x)$ тј да је f $\frac{\pi}{2}$-периодична, па тога одмах добијамо $I_1 = I_2 = I_3 = I_4$ и зато је $I = 4 I_1 + I_5$.

3. а) АК: $|a_n| = \left|(-1)^n\left(\sin\frac{1}{\sqrt{n}} + \ln\frac{2n+5}{2n+3}\right)\right| \overset{(*)}{=} \sin\frac{1}{\sqrt{n}} + \ln\frac{2n+5}{2n+3}$

$(*)$: $\frac{1}{\sqrt{n}} \in [0,1]$ и $\sin x \geq 0$ на $[0,1] \Rightarrow \sin\frac{1}{\sqrt{n}} \geq 0 \quad \forall n \in \mathbb{N}$

$\frac{2n+5}{2n+3} > 1 \Rightarrow \ln\frac{2n+5}{2n+3} > 0 \quad \forall n \in \mathbb{N}$

$\sin\underset{\downarrow 0}{\frac{1}{\sqrt{n}}} \sim \frac{1}{\sqrt{n}}, \ n\to\infty$ и $\ln\frac{2n+5}{2n+3} = \ln\left(1+\underset{\downarrow 0}{\frac{2}{2n+3}}\right) \sim \frac{2}{2n+3} =$

$$= \frac{2}{n\left(2+\underset{\downarrow 0}{\frac{3}{n}}\right)} \sim \frac{2}{n\cdot 2} = \frac{1}{n}, \ n\to\infty$$

$\Rightarrow |a_n| \sim \frac{1}{\sqrt{n}} + \frac{1}{n} = \frac{1}{\sqrt{n}}\left(1+\underset{\downarrow 0}{\frac{1}{\sqrt{n}}}\right) \sim \frac{1}{\sqrt{n}}\cdot 1 = \frac{1}{n^{1/2}}, \ n\to\infty$

$\frac{1}{2} < 1 \overset{\text{II п.к.}}{\Rightarrow} \sum_{n=1}^{\infty} a_n$ апс. дивергира

УК: $b_n = \sin\frac{1}{\sqrt{n}} + \ln\frac{2n+5}{2n+3} = \sin\frac{1}{\sqrt{n}} + \ln\left(1+\frac{2}{2n+3}\right) > 0$

$\frac{1}{2} < 1 \overset{\text{II п.к.}}{\Rightarrow} \sum_{n=1}^{\infty} a_n$ апс. дивергира

УК: $\quad b_n = \sin\frac{1}{\sqrt{n}} + \ln\frac{2n+5}{2n+3} = \sin\frac{1}{\sqrt{n}} + \ln\left(1+\frac{2}{2n+3}\right) > 0$

$\frac{1}{\sqrt{n}} \searrow$ и $\sin x \nearrow$ на $[0,1] \Rightarrow \sin\frac{1}{\sqrt{n}} \searrow$

$(2n+3)\nearrow \Rightarrow \left(1+\frac{2}{2n+3}\right)\searrow$ и $\ln x \nearrow \Rightarrow \ln\left(1+\frac{2}{2n+3}\right)\searrow$

$\Rightarrow b_n \searrow$ као збир два опадајућа низа

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\sin\underbrace{\frac{1}{\sqrt{n}}}_{\to 0} + \ln\left(1+\underbrace{\frac{2}{2n+3}}_{\to 0}\right)\right) = \sin 0 + \ln 1 = 0$

$\overset{\text{Лајбниц}}{\Rightarrow} \sum_{n=1}^{\infty}\underbrace{(-1)^n b_n}_{a_n}$ (К), па и (УК)

б) А1) $\sum_{n=1}^{\infty} a_n$ (К) ✓

А2) $b_n = \operatorname{arctg}\left(\sin\frac{n+23}{2n+2023}\right)$

b_n је ограничен, јер $|b_n| < \frac{\pi}{2}$ (због arctg)

$$\frac{n+23}{2n+2023} = \frac{n+\frac{2023}{2}-\frac{2023}{2}+23}{2n+2023} = \frac{1}{2} - \frac{1977}{2n+2023}$$

опада

$$\Rightarrow \left(\frac{n+23}{2n+2023}\right) \nearrow$$

такође, $\frac{n+23}{2n+2023} \to \frac{1}{2},\ n\to\infty$

$\Rightarrow \frac{n+23}{2n+2023} \in (0,1)$ за довољно велико $n \geq n_0$

$\sin x \nearrow$ на $(0,1)$ $\Rightarrow$ $\sin\frac{n+23}{2n+2023} \nearrow$ почев од неког n_0

$\operatorname{arctg} x \nearrow$ на $\mathbb{R}$ $\Rightarrow$ $\underbrace{\operatorname{arctg}\left(\sin\frac{n+23}{2n+2023}\right)}_{=b_n} \nearrow$ од n_0

Абел $\Rightarrow$ $\sum_{n=1}^{\infty} a_n b_n$ (К)

пронашли

b_n

Абел $\Rightarrow$ $\sum_{n=1}^{\infty} a_n b_n$ (K)

4. сума коначног геометријског низа

$$\frac{1-c^{2023}}{1-c} = 1+c+c^2+\dots+c^{2022}$$

? $(\exists c\in(0,1))$ $f(c)=1+c+c^2+\dots+c^{2022}$

тј. показујемо да фја $f(x)-1-x-x^2-\dots-x^{2022}$

има нулу на $(0,1)$

Дефинишемо $F(x)=\int_0^x f(t)\,dt$, $F:[0,1]\to\mathbb{R}$

примитивну фју фје f на $[0,1]$

$\Rightarrow F'(x)=f(x) \quad \forall x\in(0,1)$

и дефинишемо $g(x)=F(x)-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots-\frac{x^{2023}}{2023}$

$g:[0,1]\to\mathbb{R}$ непр. и диф. на $(0,1)$ као композиција

$$g(0) = F(0) - 0 - 0 - \dots - 0 = F(0) = \int_0^0 f(t)dt = 0$$

$$g(1) = F(1) - 1 - \frac{1}{2} - \dots - \frac{1}{2023} = \int_0^1 f(t)dt - 1 - \frac{1}{2} - \dots - \frac{1}{2023} =$$

$$= 1 + \frac{1}{2} + \dots + \frac{1}{2023} - 1 - \frac{1}{2} - \dots - \frac{1}{2023} = 0 = g(0)$$

$$\overset{\text{Рол}}{\Rightarrow} (\exists c \in (0,1))\ g'(c) = 0$$

$$g'(x) = \left(F(x) - x - \frac{x^2}{2} - \dots - \frac{x^{2023}}{2023}\right)' = f(x) - 1 - x - \dots - x^{2022}$$

$\Rightarrow$ c је нула фје $F(x) - 1 - x - \dots - x^{2022}$ што је и требало доказати