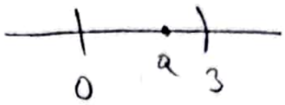
1. а) $\sup A = 3$?

1) $(\forall a \in A)\ a \le 3$ ✓

2) $(\forall \varepsilon > 0)(\exists a \in A)\ a > 3 - \varepsilon$ ✓

$\varepsilon > 0$ произв.



за a можемо узети било који позитиван број интервала $(3-\varepsilon, 3)$

$\Rightarrow \sup A = 3$

б) $B = \{5x \mid x \in A\} = [5, 15)$

и као у а) лако закључујемо $\sup B = 15$

в) означимо $s = \sup C$

$\Rightarrow$ i) $(\forall a \in C)\ a \le s$

ii) $(\forall \varepsilon > 0)(\exists a \in C)\ a > s - \varepsilon$

покажимо $\sup\{\mu a \mid a \in C\} = \mu s$:

1) $(\forall a \in C)\ \mu a \le \mu s$?

из i) је $(\forall a \in C)\ a \le s$, па множењем обе стране са $\mu \ge 0$ имамо $(\forall a \in C)\ \mu a \le \mu s$ ✓

2) $(\forall \varepsilon > 0)(\exists a \in C)\ \mu a > \mu s - \varepsilon$?

$\varepsilon > 0$ произв.

1° $\mu = 0$

за a можемо узети било који елемент из C, тривијално важи

$$\underbrace{0 \cdot a}_{=0} > 0 \cdot s - \varepsilon = -\varepsilon$$

2° $\mu > 0$

$\Rightarrow \frac{\varepsilon}{\mu} > 0$

$\overset{(ii)}{\Rightarrow} (\exists a \in C)\ a > s - \frac{\varepsilon}{\mu} \quad | \cdot \mu$

$\Rightarrow (\exists a \in C)\ \mu a > \mu s - \varepsilon \quad \checkmark$

$\Rightarrow \sup\{\mu a \mid a \in C\} = \mu s$

в) Не важи. Нпр. $\mu = -1$ и $C = \{1,2\}$.

$$\sup\{\mu a \mid a \in C\} = \sup\{-1,-2\} = -1 \neq -2 = -1 \cdot 2 = \mu \sup C$$

г) Упоређујемо a_n и a_{n+1}:

$$a_n \ \square \ a_{n+1}$$

$$\frac{n^2+1}{2n^2+n} \ \square \ \frac{(n+1)^2+1}{2(n+1)^2+n+1}$$

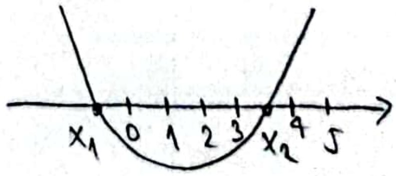
$$\frac{n^2+1}{2n^2+n} \ \square \ \frac{n^2+2n+2}{2n^2+5n+3} \quad \Big| \cdot (2n^2+n)(2n^2+5n+3) > 0$$

$$\cancel{2n^4} + \cancel{5n^3} + 3n^2 + 2n^2 + 5n + 3 \ \square \ \cancel{2n^4} + \cancel{n^3} + \cancel{4n^3} + 2n^2 + 4n^2 + 2n$$

$$3n + 3 \ \square \ n^2$$

$x^2 - 3x - 3 = 0$

$x_{1,2} = \frac{3 \pm \sqrt{21}}{2}$



$x_1 = \frac{3-\sqrt{21}}{2} < 0 \qquad x_2 = \frac{3+\sqrt{21}}{2} < \frac{3+5}{2} = 4 \ , \ x_2 = \frac{3+\sqrt{21}}{2} > \frac{3+4}{2} = \frac{7}{2}$

$\Rightarrow$ за $n \in \{1,2,3\}$ је $n^2-3n-3<0$ тј. $3n+3>n^2$,

а за $n \geq 4$ је $n^2-3n-3>0$ тј. $3n+3<n^2$

$\Rightarrow$ $a_n > a_{n+1}$ за $n \in \{1,2,3\}$ и

$a_n < a_{n+1}$ за $n \geq 4$

$$\underbrace{a_1 > a_2 > a_3 > a_4}_{\text{опада}} \quad \underbrace{a_4 < a_5 < a_6 < a_7 < a_8 < a_9 < \ldots}_{\text{расте}}$$

π)

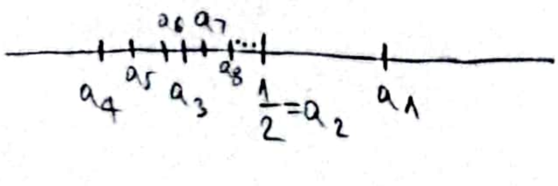
$$\sup D = \sup\left\{\pi \cdot \sin\frac{n\pi}{2} \cdot \frac{n^2+1}{2n^2+n} \,\middle|\, n \in \mathbb{N}\right\} \overset{б)}{=} \pi \cdot \underbrace{\sup\left\{\sin\frac{n\pi}{2} \cdot \frac{n^2+1}{2n^2+n} \,\middle|\, n \in \mathbb{N}\right\}}_{=:E}$$

$$\sin\frac{n\pi}{2} = \begin{cases} 1, & n=4k+1 \\ 0, & n=4k+2 \text{ или } n=4k \\ -1, & n=4k+3 \end{cases}$$

$a_1 = \frac{2}{3}$, $a_2 = \frac{5}{10} = \frac{1}{2}$, $a_3 = \frac{10}{21}$, $a_4 = \frac{17}{36}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{2n^2+n} \overset{:n^2}{=} \frac{1}{2} < \frac{2}{3} = a_1$$

$\Rightarrow$ $a_n \in \left[\frac{17}{36}, \frac{1}{2}\right)$ за $n \geq 4$ (јер расте почев од 4. члана и тежи ка $\frac{1}{2}$)



$\Rightarrow$ највећи члан низа (a_n) је $a_1 = \frac{2}{3}$

$$E = \left\{ \underbrace{\sin\frac{n\pi}{2}}_{\in\{-1,0,1\}} \cdot \underbrace{\frac{n^2+1}{2n^2+n}}_{\le \frac{2}{3}} \;\middle|\; n\in\mathbb{N} \right\}$$

$\Rightarrow (\forall x\in E)\ x \le 1\cdot\frac{2}{3} = \frac{2}{3}$, па је $\frac{2}{3}$ горње огр. скупа E

Такође, $\frac{2}{3}\in E$ $\left(\sin\frac{\pi}{2} = 1 \text{ и } \frac{1^2+1}{2\cdot 1^2+1} = \frac{2}{3}\right)$.

$\Rightarrow \quad \sup E = \max E = \frac{2}{3}$

$\Rightarrow \quad \sup D = \max D = \frac{2\pi}{3}$

2\. а) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{8n^6-3n^4+n^2-n+1}{4n^6+n^5-n^4+2n^2} \overset{:n^6}{=} \dfrac{8}{4} = 2$

б) $c_n = \sqrt[n]{3^n + 4^{(-1)^n\cdot n}}$

због $(-1)^n$ посматрамо поднизове c_{2k} и c_{2k+1}:

$c_{2k} = \sqrt[2k]{3^{2k}+4^{2k}} \to \max\{3,4\} = 4,\ k\to\infty$

$c_{2k+1} = \sqrt[2k+1]{3^{2k+1}+4^{-(2k+1)}} = \sqrt[2k+1]{3^{2k+1}+\left(\frac{1}{4}\right)^{2k+1}} \to$

$\to \max\{3,\frac{1}{4}\} = 3,\ k\to\infty$

$\Rightarrow$ тачке нагомилавања низа (c_n) су 4 и $\frac{1}{3}$

$\Rightarrow T = \limsup\limits_{n\to\infty} c_n = 4$

b) $a_n b_n - \left(\cos\frac{1}{n}\right)^{n^2} = T$

$$\Rightarrow b_n = \frac{T + \left(\cos\frac{1}{n}\right)^{n^2}}{a_n} = \frac{4 + \left(\cos\frac{1}{n}\right)^{n^2}}{a_n}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{4+\left(\cos\frac{1}{n}\right)^{n^2}}{a_n} = \frac{4+\lim\limits_{n\to\infty}\left(\cos\frac{1}{n}\right)^{n^2}}{\lim\limits_{n\to\infty} a_n} =$$

$$= \frac{4+\lim\limits_{n\to\infty} e^{n^2\ln\left(\cos\frac{1}{n}\right)}}{2} = 2+\frac{1}{2}\lim_{n\to\infty} e^{n^2\cdot\ln\left(1-\frac{\frac{1}{n^2}}{2}+o\left(\frac{1}{n^2}\right)\right)}$$

$$= 2+\frac{1}{2}\lim_{n\to\infty} e^{n^2\cdot\ln\left(1+\left(-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)\right)} =$$

$$= 2+\frac{1}{2}\lim_{n\to\infty} e^{n^2\cdot\left(-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)} = 2+\frac{1}{2}\lim_{n\to\infty} e^{-\frac{1}{2}+o(1)}$$

$$= 2+\frac{1}{2}e^{-\frac{1}{2}} = 2+\frac{1}{2\sqrt{e}}$$

3. $a_1 = a \geq 0, \quad a_{n+1} = a_n + 8\cdot\sqrt[4]{a_n^3}, \quad n\in\mathbb{N}$

1° $a=0 \Rightarrow a_n = 0 \quad \forall n\in\mathbb{N}$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

2° $a>0 \Rightarrow a_1>0$ (база)

индукцијом $a_n>0 \quad \forall n\in\mathbb{N}$

$a_n>0 \overset{?}{\Rightarrow} a_{n+1}>0$

$a_{n+1} = \underbrace{a_n}_{>0} + 8\cdot\underbrace{\sqrt[4]{a_n^3}}_{>0} > 0$ ✓

$a_{n+1}-a_n = 8\cdot\sqrt[4]{a_n^3} > 0$

$\Rightarrow a_{n+1} > a_n \quad \forall n\in\mathbb{N}$

$\Rightarrow (a_n) \nearrow$ строго

$\Rightarrow$ i) (a_n) је огр. одозго

$$\Downarrow \text{T.}$$

(a_n) конв.

$$\Downarrow$$

$$\exists \lim_{n\to\infty} a_n = L \in \mathbb{R}$$

$$a_{n+1} = a_n + 8\cdot\sqrt[4]{a_n^3} \;\Big/ \lim_{n\to\infty}$$

$$L = L + 8\cdot\sqrt[4]{L^3}$$

$$L = 0 \quad ↯$$

$(a_1 > 0$ и $(a_n) \nearrow$ строго$)$

ii) (a_n) није огр. одозго

$$\Downarrow$$

$$\boxed{\lim_{n\to\infty} a_n = +\infty}$$

ово важи следеће

б) Штолц ($n\nearrow$ и тежи $+\infty$)

$$\lim_{n\to\infty} \frac{\sqrt[4]{a_n}}{n} \overset{?}{=} \lim_{n\to\infty} \frac{\sqrt[4]{a_{n+1}} - \sqrt[4]{a_n}}{n+1-n} =$$

$$= \lim_{n\to\infty} \Big(\underbrace{\sqrt[4]{a_n + 8\sqrt[4]{a_n^3}}}_{\text{дати}} - \sqrt[4]{a_n} \Big) = \lim_{n\to\infty} \Big(\sqrt[4]{a_n\Big(1+\frac{8}{\sqrt[4]{a_n}}\Big)} - \sqrt[4]{a_n} \Big) =$$

$$= \lim_{n\to\infty} \Big(\sqrt[4]{a_n}\cdot\Big(1+\underbrace{\frac{8}{\sqrt[4]{a_n}}}_{\to 0}\Big)^{\frac14} - \sqrt[4]{a_n} \Big) = \Big((1+x)^\alpha = 1+\alpha x + o(x),\ x\to 0 \Big)$$

$$= \lim_{n\to\infty} \sqrt[4]{a_n}\Big(1 + \frac14\cdot\frac{8}{\sqrt[4]{a_n}} + o\Big(\frac{1}{\sqrt[4]{a_n}}\Big) - 1\Big) =$$

$$= \lim_{n\to\infty} \big(2 + o(1)\big) = 2$$

в)

$$\lim_{n\to\infty} \Big(\frac{a_{n+1}}{a_n}\Big)^n = \lim_{n\to\infty} \Big(\frac{a_n + 8\sqrt[4]{a_n^3}}{a_n}\Big)^n = \lim_{n\to\infty} \Big(1+\frac{8}{\sqrt[4]{a_n}}\Big)^n =$$

$$= \lim_{n\to\infty} e^{n\cdot \ln\left(1+\underset{\downarrow 0}{\frac{8}{\sqrt[4]{a_n}}}\right)} = \lim_{n\to\infty} e^{n\cdot\left(\frac{8}{\sqrt[4]{a_n}} + o\left(\frac{1}{\sqrt[4]{a_n}}\right)\right)} =$$

$$= \lim_{n\to\infty} e^{8\cdot \frac{n}{\sqrt[4]{a_n}} + o\left(\frac{n}{\sqrt[4]{a_n}}\right)} \overset{б)}{=} \lim_{n\to\infty} e^{8\cdot\frac{1}{2} + o\left(\frac{1}{2}\right)} = e^4$$

в) $$\lim_{n\to\infty} n^4 a_n \left(2 + \sin^2\underset{\downarrow 0}{\left(\frac{2}{a_n}\right)} - \sqrt[3]{8 - \frac{9}{a_n^2}}\right) =$$

$$= \lim_{n\to\infty} n^4 a_n \left(2 + \left(\frac{2}{a_n} + o\left(\frac{1}{a_n}\right)\right)^2 - \sqrt[3]{8}\cdot\sqrt[3]{1-\frac{9}{8a_n^2}}\right) =$$

$$= \lim_{n\to\infty} n^4 a_n \left(2 + \frac{4}{a_n^2} + o\left(\frac{1}{a_n^2}\right) - 2\cdot\left(1 + \underset{\to 0}{\frac{-9}{8a_n^2}}\right)^{\frac{1}{3}}\right) =$$

$$= \lim_{n\to\infty} n^4 a_n \left(2 + \frac{4}{a_n^2} + o\left(\frac{1}{a_n^2}\right) - 2\left(1 + \frac{1}{3}\cdot\frac{-9}{8a_n^2} + o\left(\frac{1}{a_n^2}\right)\right)\right) =$$

$$= \lim_{n\to\infty} n^4 a_n \left(\cancel{2} + \frac{\overset{\cdot 4}{4}}{a_n^2} + o\left(\frac{1}{a_n^2}\right) - \cancel{2} + \frac{3}{4a_n^2} + o\left(\frac{1}{a_n^2}\right)\right) =$$

$$= \lim_{n\to\infty} n^4 a_n \left(\frac{19}{4a_n^2} + o\left(\frac{1}{a_n^2}\right)\right) = \lim_{n\to\infty}\left(\frac{19}{4}\cdot\frac{n^4}{a_n} + o\left(\frac{n^4}{a_n}\right)\right) =$$

$$\overset{б)}{=} \lim_{n\to\infty}\left(\frac{19}{4}\cdot\left(\frac{1}{2}\right)^4 + o\left(\frac{1}{2}\right)\right) = \frac{19}{64} \quad \left(\frac{n^4}{a_n} \to \frac{1}{16},\ n\to\infty,\ \text{јер } \frac{n}{\sqrt[4]{a_n}} \to \frac{1}{2},\ n\to\infty\right)$$

д) T - период од f

f ограничена на $[0,T]$ по Вајерштрасу

$\Rightarrow (\exists M>0)(\forall x\in[0,T])\ |f(x)| \le M$

$\overset{T\text{-периодична}}{\Rightarrow} (\forall x\in\mathbb{R})\ |f(x)| \le M \Rightarrow f$ је ограничена

$$\Rightarrow \lim_{n\to\infty} \frac{\overset{\text{огр.}}{f(e^{a_n})}}{\underset{\to+\infty}{a_n}} = 0$$

4. f је непр. на $(-\infty,-2)\cup(-2,0)\cup(0,1]$ као композиција непрекидних фја (свака грана је непр. фја)

Још треба испитати тачке -2 и 0.

$\underline{x_0=-2:}$

$$\lim_{x\to-2-} f(x) = f(-2) = \lim_{x\to-2+} f(x)$$

$$\lim_{x\to-2-} f(x) = \lim_{x\to-2-} \frac{\cos 3\pi x + e^{2x+3}+1}{x^2-2} + a$$

$$= \frac{\cos(-6\pi)+e^{-1}+1}{4-2}+a = \frac{\overset{1}{\cos 0}+\frac{1}{e}+1}{2}+a = 1+\frac{1}{2e}+a$$

$$f(-2) = \sqrt[3]{-2+1}\cdot|4-2| = (-1)\cdot 2 = -2$$

$$\lim_{x\to-2+} f(x) = \lim_{x\to-2+} \sqrt[3]{x+1}\cdot|x^2+x| = -1\cdot 2 = -2$$

$$\Rightarrow 1+\frac{1}{2e}+a = -2 = -2$$

$$\Rightarrow \boxed{a = -3-\frac{1}{2e}}$$

$\underline{x_0=0:}$

$$\lim_{x\to0-} f(x) = f(0) = \lim_{x\to0+} f(x)$$

$$f(0) = \sqrt[3]{1}\cdot 0 = 0$$

$$\lim_{x\to0-} f(x) = \lim_{x\to0-} \sqrt[3]{x+1}\cdot|x^2+x| = \sqrt[3]{0+1}\cdot|0+0| = 0$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\cos 3x + e^{x^2} - 2}{\ln(1+5x)\sin 2x} + b =$$

$$= \lim_{x\to 0+} \frac{\cancel{1} - \frac{(3x)^2}{2} + o(x^2) + \cancel{1} + x^2 + o(x^2) - \cancel{2}}{(5x + o(x))(2x + o(x))} + b =$$

$$= \lim_{x\to 0+} \frac{-\frac{9}{2}x^2 + x^2 + o(x^2)}{10x^2 + o(x^2)} + b = \lim_{x\to 0+} \frac{-\frac{7}{2}x^2 + o(x^2)}{10x^2 + o(x^2)} \, {}^{| : x^2} + b =$$

$$= \lim_{x\to 0+} \frac{-\frac{7}{2} + \cancel{o(1)}^{\to 0}}{10 + \cancel{o(1)}^{\to 0}} + b = -\frac{7}{20} + b$$

$$\Rightarrow \quad 0 = 0 = -\frac{7}{20} + b \quad \Rightarrow \boxed{b = \frac{7}{20}}$$