VEKTORSKA ALGEBRA I ANALIZA

- skalar = brojevi
- vektor $\vec{a} = (a_1 \; a_2 \; a_3)$
- tenzori

⊗ reprezentacija

- skalar $[\;]_{1\times1}$
- vektore $[\quad]_{1\times3}$ ili $[\;]_{3\times1}$
- tenzori $[\quad]_{3\times3}$

3D Dekartovom sistemu

$$\hat{T} = \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}$$

• <u>def</u>

1) skalarni proizvod $\vec{a}, \vec{b}$

$$\vec{a} \cdot \vec{b} = \underset{\underline{1 \times 3}}{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}} \underset{\underline{3 \times 1}}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \underset{1 \times 1}{a_1 b_1 + a_2 b_2 + a_3 b_3}$$

2) vektorski proizvod $\vec{a}, \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \begin{array}{l} \leftarrow \vec{e}_x \\ \leftarrow \vec{e}_y \\ \leftarrow \vec{e}_z \end{array}$$

3) tenzorski (dijadski) proizvod $\vec{a}, \vec{b}$

(komponente $a_1 b_1$ "klasično množenje")

! $\underline{\underline{\vec{a}\vec{b}}} = \vec{a} \underline{\underline{\otimes}} \vec{b}$

$$\vec{a} \otimes \vec{b} = \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\underline{\underline{3\times 1}}} \underbrace{\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}}_{\underline{\underline{1\times 3}}} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}_{3\times 3}$$

$$\otimes \quad \underbrace{\hat{T}}_{3\times 3} \cdot \vec{v} = \underbrace{\begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} T_{11} v_1 + T_{21} v_2 + T_{31} v_3 \\ T_{12} v_1 + T_{22} v_2 + T_{32} v_3 \\ T_{13} v_1 + T_{23} v_2 + T_{33} v_3 \end{bmatrix}}_{3\times 1}$$

miro

$$\circledast \quad \underset{\substack{= \\ 1\times 3}}{\vec{v}} \cdot \underset{3\times 3}{\hat{T}} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} = \overset{1\times 3}{\begin{bmatrix} v_1 T_{11} + v_2 T_{12} + v_3 T_{13} ; \\ v_1 T_{21} + v_2 T_{22} + v_3 T_{23} ; \\ v_1 T_{31} + v_2 T_{32} + v_3 T_{33} \end{bmatrix}}$$

$$\circledast \quad \hat{T} = \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ \underbrace{T_{13}}_{\vec{T}_1} & \underbrace{T_{23}}_{\vec{T}_2} & \underbrace{T_{33}}_{\vec{T}_3} \end{bmatrix} = \left[\vec{T}_1 , \vec{T}_2 , \vec{T}_3 \right]$$

$\circledast$ operator nabla ∇ :

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix}$$

domaći: pokazati (pomoću matrica): Ako je $\hat{T}$ simetričan, tada važi $\vec{v} \cdot \hat{T} = \hat{T} \cdot \vec{v}$.

$$\hat{T} = \begin{bmatrix} T_1 & T_4 & T_5 \\ T_4 & T_2 & T_6 \\ T_5 & T_6 & T_3 \end{bmatrix}$$

(*) def: 1) skalarno polje: $T(\vec{r}, t)$

$$dT = d\vec{r} \cdot (\nabla T)$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{v} \cdot (\nabla T)$$

2) vektorsko polje: $\underline{\vec{A}}(\vec{r}, t)$

$$d\vec{A} = d\vec{r} \cdot (\nabla \otimes \vec{A})$$

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + \vec{v} \cdot \underbrace{(\nabla \otimes \vec{A})}_{\text{lokalni tenzor vekt. polja}}$$

- disperzija

$$\vec{v} \cdot (\nabla T) = \underset{1\times 3}{[\quad]} \left(\underset{3\times 1}{[\;]} \underset{1\times 1}{[\;]} \right) = \underset{1\times 3}{[\quad]} \underset{3\times 1}{[\;]} = \underset{1\times 1}{[\;]}$$

$$\vec{v} \cdot (\nabla \otimes \vec{A}) = [\quad] \left([\;] [\quad] \right) = [\quad] [\quad] =$$

$$= [\quad]$$

1. $\nabla \cdot (\vec{A} \otimes \vec{B}) = ?$

$$\vec{A} \otimes \vec{B} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \begin{bmatrix} B_x & B_y & B_z \end{bmatrix} = \begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix}$$

$$\nabla \cdot (\vec{A} \otimes \vec{B}) = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} \begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix} =$$

$$= \Big[\partial_x (A_x B_x) + \partial_y (A_y B_x) + \partial_z (A_z B_x) ;$$
$$\partial_x (A_x B_y) + \partial_y (A_y B_y) + \partial_z (A_z B_y) ;$$
$$\partial_x (A_x B_z) + \partial_y (A_y B_z) + \partial_z (A_z B_z) \Big]$$

miro

$$\left.\begin{aligned}
&\partial_x(A_xB_x) + \partial_y(A_yB_x) + \partial_z(A_zB_x) = B_x\left(\partial_x A_x + \partial_y A_y + \partial_z A_z\right) + \\
&\qquad + A_x\,\partial_x B_x + A_y\,\partial_y B_x + A_z\,\partial_z B_x = \\
&= B_x\Big(\underbrace{\partial_x A_x + \partial_y A_y + \partial_z A_z}_{\mathrm{div}\,\vec{A}}\Big) + \Big(\underbrace{A_x\partial_x + A_y\partial_y + A_z\partial_z}_{\vec{A}\cdot\nabla}\Big)B_x \\
&=
\end{aligned}\right\}\vec{e}_x$$

• slično i za 2. i 3. komponentu

$$\boxed{(\nabla\cdot\vec{A})\vec{B} + (\vec{A}\cdot\nabla)\vec{B} = \nabla\cdot(\vec{A}\otimes\vec{B})}$$

2. način: ∇ - operator

1) osobina sk. pr. $\hat{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

$$\nabla \cdot (\vec{A} \otimes \vec{B}) = \nabla \cdot (\overset{\downarrow}{\vec{A}} \otimes \vec{B}) + \nabla \cdot (\vec{A} \otimes \overset{\downarrow}{\vec{B}}) =$$

$$= (\nabla \cdot \overset{\downarrow}{\vec{A}})\vec{B} + (\vec{A} \cdot \nabla)\overset{\downarrow}{\vec{B}}$$

(2) $\nabla \times (\vec{A} \times \vec{B}) = ?$ (uvek za domaći možete raspisati)

$$\nabla \times (\vec{A} \times \vec{B}) = \nabla \times (\overset{\downarrow}{\vec{A}} \times \vec{B}) + \nabla \times (\vec{A} \times \overset{\downarrow}{\vec{B}})$$

$$\rightarrow \quad \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$$

$$\nabla \times (\overset{\downarrow}{\vec{A}} \times \vec{B}) = \overset{\downarrow}{\vec{A}} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \overset{\downarrow}{\vec{A}}) = (\vec{B} \cdot \nabla) \overset{\downarrow}{\vec{A}} - \vec{B} (\nabla \cdot \overset{\downarrow}{\vec{A}})$$

$$\nabla \times (\vec{A} \times \overset{\downarrow}{\vec{B}}) = -\nabla \times (\overset{\downarrow}{\vec{B}} \times \vec{A}) = -\left[\overset{\downarrow}{\vec{B}} (\nabla \cdot \vec{A}) - \vec{A} (\nabla \cdot \overset{\downarrow}{\vec{B}}) \right] =$$

$$\rightarrow \quad \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$= \vec{A} (\nabla \cdot \overset{\downarrow}{\vec{B}}) - (\vec{A} \cdot \nabla) \overset{\downarrow}{\vec{B}}$$

$$\boxed{\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \overset{\downarrow}{\vec{A}} - \vec{B} (\nabla \cdot \overset{\downarrow}{\vec{A}}) + \vec{A} (\nabla \cdot \overset{\downarrow}{\vec{B}}) - (\vec{A} \cdot \nabla) \overset{\downarrow}{\vec{B}}}$$

(3) $\nabla \cdot (\vec{A} \times \vec{B}) = ?$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \nabla \cdot (\vec{A}^{\downarrow} \times \vec{B}) + \nabla \cdot (\vec{A} \times \vec{B}^{\downarrow}) =$$

→ osobina mešovitog proizvoda:
invarijantan u odnosu na cikličnе permutacije:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$= \vec{B} \cdot (\nabla \times \vec{A}^{\downarrow}) - \vec{A} \cdot (\nabla \times \vec{B}^{\downarrow})$$

(4) $\nabla \times (T\vec{A}) = ?$

$$\nabla \times (T\vec{A}) = \nabla \times (T^{\downarrow}\vec{A}) + \nabla \times (T\vec{A}^{\downarrow}) =$$

$$= \nabla T^{\downarrow} \times \vec{A} + T (\nabla \times \vec{A}^{\downarrow})$$

⑤ $\nabla(\vec{A}\cdot\vec{B}) = ?$

$$\nabla(\vec{A}\cdot\vec{B}) = \nabla\underline{(\overset{\downarrow}{\vec{A}}\cdot\vec{B})} + \nabla\underline{(\vec{A}\cdot\overset{\downarrow}{\vec{B}})} = \quad \circledast$$

$$\vec{A}\times(\nabla\times\overset{\downarrow}{\vec{B}}) = \nabla(\vec{A}\cdot\overset{\downarrow}{\vec{B}}) - \overset{\downarrow}{\vec{B}}(\vec{A}\cdot\nabla) =$$

$$= \nabla\underbrace{(\vec{A}\cdot\overset{\downarrow}{\vec{B}})} - (\vec{A}\cdot\nabla)\overset{\downarrow}{\vec{B}}$$

$$\vec{B}\times(\nabla\times\overset{\downarrow}{\vec{A}}) = \nabla(\vec{B}\cdot\overset{\downarrow}{\vec{A}}) - \overset{\downarrow}{\vec{A}}(\vec{B}\cdot\nabla) =$$

$$= \nabla\underbrace{(\overset{\downarrow}{\vec{A}}\cdot\vec{B})} - (\vec{B}\cdot\nabla)\overset{\downarrow}{\vec{A}}$$

$$\nabla\left(\vec{A}\cdot\overset{\downarrow}{\vec{B}}\right) = \vec{A}\times\left(\nabla\times\overset{\downarrow}{\vec{B}}\right) + \left(\vec{A}\;\nabla\right)\overset{\downarrow}{\vec{B}}$$

$$\nabla\left(\overset{\downarrow}{\vec{A}}\cdot\vec{B}\right) = \vec{B}\times\left(\nabla\times\overset{\downarrow}{\vec{A}}\right) + \left(\vec{B}\cdot\nabla\right)\overset{\downarrow}{\vec{A}}$$

$$\overset{*}{=}\; \vec{A}\times\left(\nabla\times\overset{\downarrow}{\vec{B}}\right) + \left(\vec{A}\cdot\nabla\right)\overset{\downarrow}{\vec{B}} + \vec{B}\times\left(\nabla\times\overset{\downarrow}{\vec{A}}\right) + \left(\vec{B}\cdot\nabla\right)\overset{\downarrow}{\vec{A}}$$

6 $\quad L\;\; \underbrace{\vec{A}\left(\vec{B}\cdot\vec{C}\right)}_{\mathrm{I}} = \left(\underbrace{\vec{A}\otimes\vec{B}}_{\mathrm{II}}\right)\cdot\vec{C}$ (pokazati)

$(i = 1, 2, 3)$

I

$$\vec{A}\,(\vec{B}\cdot\vec{C}) = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \left(\left[\sum_i B_i C_i \right] \right) = \begin{bmatrix} A_x \sum_i B_i C_i \\ A_y \sum_i B_i C_i \\ A_z \sum_i B_i C_i \end{bmatrix}$$

II

$$(\vec{A} \otimes \vec{B}) \cdot \vec{C} = \left(\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \begin{bmatrix} B_x & B_y & B_z \end{bmatrix} \right) \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} =$$

$$= \begin{bmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{bmatrix} \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} =$$

$$= \begin{bmatrix} A_x B_x C_x + A_x B_y C_y + A_x B_z C_z \\ A_y B_x C_x + A_y B_y C_y + A_y B_z C_z \\ A_z B_x C_x + A_z B_y C_y + A_z B_z C_z \end{bmatrix} = \begin{bmatrix} A_x & \sum_i B_i C_i \\ A_y & \sum_i B_i C_i \\ A_z & \sum_i B_i C_i \end{bmatrix}$$

$\blacksquare$

(7) Neka je $\vec{B}$ linearna f-ja vektora $\vec{A}$ data kao:

$$\vec{B} = \vec{a} \times (\vec{b} \times \vec{A}),$$

gde su $\vec{a}$ i $\vec{b}$ poznati vektori.

Odrediti tenzor $\hat{T}$ takav da je:

$$\vec{B} = \hat{T} \cdot \vec{A}$$

$$\vec{B} = \vec{a} \times (\vec{b} \times \vec{A}) = \vec{b}\,(\vec{a} \cdot \vec{A}) - \vec{A}\,(\vec{a} \cdot \vec{b}) =$$

$$= (\vec{b} \otimes \vec{a}) \cdot \vec{A} - (\vec{a} \cdot \vec{b})\vec{A} =$$

$$= \left[(\vec{b} \otimes \vec{a}) \cdot \quad - (\vec{a} \cdot \vec{b}) \right] \vec{A} =$$

$$\cdot\,\hat{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\underbrace{\vec{b} \otimes \vec{a} - (\vec{a} \cdot \vec{b})\hat{I}}_{\hat{T}} \right] \cdot \vec{A}$$

$$\hat{T} = \begin{bmatrix} -a_2 b_2 - a_3 b_3 & b_1 a_2 & b_1 a_3 \\ b_2 a_1 & -a_1 b_1 - a_3 b_3 & b_2 a_3 \\ b_3 a_1 & b_3 a_2 & -a_1 b_1 - a_2 b_2 \end{bmatrix}$$

(8) Pokazati da je:

$$\operatorname{rot}\left(\vec{a}\,\underbrace{(\vec{b}\cdot\vec{r})}\,r^n\right) = (\vec{b}\times\vec{a})\,r^n + (\vec{r}\times\vec{a})(\vec{b}\cdot\vec{r})\,n\,r^{n-2},$$ gde je

$\vec{r}$ - vektor položaja, $r = |\vec{r}|$, $\vec{a}, \vec{b}$ su konst. vektori

$$\nabla\times(T\vec{A}) = \nabla\times(\overset{\downarrow}{T}\vec{A}) + \nabla\times(T\overset{\downarrow}{\vec{A}}) =$$

$$= \nabla\overset{\downarrow}{T}\times\vec{A} + T(\nabla\times\overset{\downarrow}{\vec{A}}) =$$

$$= -\vec{A}\times\nabla\overset{\downarrow}{T} + T(\nabla\times\vec{A})$$

miro

$$\operatorname{rot}\left(\vec{a}(\vec{b}\cdot\vec{r})r^n\right) = \underbrace{-\vec{a}\times\nabla\left((\vec{b}\cdot\vec{r})r^n\right)} + (\vec{b}\cdot\vec{r})r^n \underbrace{(\nabla\times\vec{a})}_{\operatorname{rot}(\overrightarrow{const}) = \vec{0}}$$

$$\nabla(TP) = P\nabla T + T\nabla P$$

$$\nabla\left((\vec{b}\cdot\vec{r})r^n\right) = (\vec{b}\cdot\vec{r})\,\underline{\underline{\nabla}}\, r^n + r^n\, \underline{\underline{\nabla(\vec{b}\cdot\vec{r})}}$$

$2x \cdot \frac{n}{2}$ ст. $\frac{n-2}{2}$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\nabla(r^n) = \nabla\left((x^2+y^2+z^2)^{\frac{n}{2}}\right) = \begin{bmatrix}\partial_x \\ \partial_y \\ \partial_z\end{bmatrix}\left[(x^2+y^2+z^2)^{n/2}\right] =$$

$$= \underline{\underline{n}}\; \underline{\underline{r^{n-2}}}\; \underline{\underline{\vec{r}}}$$

miro

$$\nabla(\vec{b}\cdot\vec{r}) = \nabla(b_x x + b_y y + b_z z) = \vec{b}$$

$$\mathrm{rot}\left(\vec{a}(\vec{b}\cdot\vec{r})r^n\right) = -\vec{a}\times\left[(\vec{b}\cdot\vec{r})\, n r^{n-2}\vec{r} + r^n\vec{b}\right] =$$

$$= (\vec{r}\times\vec{a})(\vec{b}\cdot\vec{r})\, n r^{n-2} + (\vec{b}\times\vec{a})\, r^n \quad \blacksquare$$