Idealna magnetohidrodinamika (MHD)

- plazma $\rightarrow$ elektroprovodni fluid

  ! procesi na prostornim i vremenskim skalama $\gg$ srednja slob. putanja čestica

$\Rightarrow$ plazma u spolj. $\vec{B}$ na $\uparrow$ prost. i vrem. skal. i spori procesi

- Ojlerov pristup $\approx$ teorija polja $\Rightarrow \vec{u}(\vec{r},t)$; $\rho(\vec{r},t)$; $p(\vec{r},t)$; $T(\vec{r},t)$; $\vec{B}(\vec{r},t)$

- model MEH. i TD. idealni fluid

  $\hat{P} \rightarrow p$; nema disipativnih procesa

⊗ Sistem jdna: jdna kont; jdna kretanja; jdna promene unutr. energije; jdna indukcije ⊕ $\nabla \cdot \vec{B} = 0$

8 skalarnih dif. jdna za 8 skalarnih promenljivih $\rho, \vec{v}, \vec{B}, e$

⊗ u formi ZO: ZOM, ZOI, ZOE, ZOMF

① Polazeći od makroskopskih jdna itresti jdne MHD u formi ZO.

1) $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$

2) $\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla\cdot(\rho\vec{v}\otimes\vec{v}) = -\nabla\cdot\hat{P} + \vec{j}\times\vec{B}$

3) $\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e + \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right) +$

$+ \nabla\cdot\left(\left(\frac{1}{2}\rho u^2 + \rho e\right)\vec{u} + \vec{u}\cdot\hat{P} + \vec{q} + \vec{S}\right) = 0$

4) $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

1) def: $\vec{\pi} = \rho\vec{v}$ $\Rightarrow$ $\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{\pi} = 0}$

2) $\frac{\partial}{\partial t}\vec{\pi} + \nabla\cdot(\underbrace{\rho\vec{v}\otimes\vec{v} + \hat{P}}_{\circledast}) \underbrace{- \vec{j}\times\vec{B}}_{\circledast} = 0$

$\circledast$ $\rho\vec{v}\otimes\vec{v} + \hat{P} \longrightarrow \rho\vec{v}\otimes\vec{v} + p\alpha\hat{I}$

$\circledast$ $\vec{j}\times\vec{B} = \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B}$ $\textcircled{=}$

$\nabla\cdot(\vec{A}\otimes\vec{B}) = \vec{B}(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla\otimes\vec{B})$

$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$\nabla\cdot(\vec B\otimes\vec B) = \vec B\underbrace{(\nabla\cdot\vec B)}_{\to 0} + \vec B\cdot(\nabla\otimes\vec B)$$

$$= \frac{1}{\mu_0}\overset{\downarrow}{\vec B}(\vec B\cdot\nabla) - \frac{1}{\mu_0}\nabla(\vec B\cdot\overset{\downarrow}{\vec B})$$

$$= \frac{1}{\mu_0}(\vec B\cdot\nabla)\overset{\downarrow}{\vec B} - \frac{1}{\mu_0}\nabla(\overset{\downarrow}{\vec B}\cdot\vec B)$$

$$= \frac{1}{\mu_0}\vec B\cdot(\nabla\otimes\overset{\downarrow}{\vec B}) - \frac{1}{\mu_0}\nabla(\overset{\downarrow}{\vec B}\cdot\vec B) =$$

$$\nabla(\vec B\cdot\vec B) = \nabla(\overset{\downarrow}{\vec B}\cdot\vec B) + \nabla(\vec B\cdot\overset{\downarrow}{\vec B}) =$$

$$= 2\nabla(\overset{\downarrow}{\vec B}\cdot\vec B)$$

$$\nabla(\overset{\downarrow}{\vec B}\cdot\vec B) = \nabla\left(\frac{B^2}{2}\right)$$

$$= \frac{1}{\mu_0}\nabla\cdot(\vec B\otimes\vec B) - \frac{1}{\mu_0}\nabla\left(\frac{B^2}{2}\right) =$$

$$= \frac{1}{\mu_0}\nabla\cdot(\vec B\otimes\vec B) - \nabla\cdot\left(\frac{B^2}{2\mu_0}\hat{I}\right)$$

$$= \nabla\cdot\left(\frac{1}{\mu_0}\vec B\otimes\vec B - \frac{B^2}{2\mu_0}\hat{I}\right)$$

$$\Rightarrow \frac{\partial \vec{\pi}}{\partial t} + \nabla\cdot\left(\rho\vec{v}\otimes\vec{v} + p_\alpha \hat{I}\right) - \nabla\cdot\left(\frac{1}{\mu_0}\vec{B}\otimes\vec{B} - \frac{B^2}{2\mu_0}\hat{I}\right) = 0$$

$$\frac{\partial \vec{\pi}}{\partial t} + \nabla\cdot\underbrace{\left(\rho\vec{v}\otimes\vec{v} + \left(p_a + \frac{B^2}{2\mu_0}\right)\hat{I} - \frac{1}{\mu_0}\vec{B}\otimes\vec{B}\right)}_{\text{def } \hat{T}} = 0$$

$$\boxed{\frac{\partial \vec{\pi}}{\partial t} + \nabla\cdot\hat{T} = \vec{0}}$$

3) $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e + \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right) +$$
$$+ \nabla\cdot\left(\left(\frac{1}{2}\rho u^2 + \rho e\right)\vec{u} + \vec{u}\cdot\hat{P} + \vec{q} + \vec{S}\right) = 0$$

a) $\rho e = \frac{3}{2}p$ → koristili smo kod jdne prenosa energije

! pps : $\gamma_g = \frac{5}{2}$ (br. step. slob 3)

⇒ u opštem slučaju:

$$\rho e = \frac{p}{\gamma_g - 1}$$

b) MHD: $\frac{1}{2}\varepsilon_0 E^2 \ll \frac{1}{2\mu_0}B^2$

c) id. MHD: nema disp. proc, nema topl. provođenja

$$\vec{g} \to 0$$

d) $\hat{p} \Rightarrow p$

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma_g - 1} + \frac{1}{2\mu_0}B^2\right)}_{\text{def } H} + \nabla\cdot\left(\underbrace{\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma_g-1}\right)\vec{u} + \vec{u}\cdot p\hat{I} + \frac{1}{\mu_0}\vec{E}\times\vec{B}}_{\text{def } \vec{U}}\right) = 0$$

$$\boxed{\vec{E} = -\vec{u}\times\vec{B} \quad ; \quad \vec{u}\cdot\hat{I} = \vec{u}}$$

$$\vec{U} = \left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma_g - 1} + p\right)\vec{u} - \frac{1}{\mu_0}\left(\vec{u}\times\vec{B}\right)\times\vec{B} =$$

$$= \left(\frac{1}{2}\rho u^2 + \frac{\gamma_g}{\gamma_g - 1}p\right)\vec{u} + \frac{1}{\mu_0}\vec{u}\left(\vec{B}\cdot\vec{B}\right) - \frac{1}{\mu_0}\vec{B}\left(\vec{B}\cdot\vec{u}\right)$$

$$= \left(\frac{1}{2}\rho u^2 + \frac{\gamma_g}{\gamma_g - 1}p\right)\vec{u} + \frac{1}{\mu_0}B^2\vec{u} - \vec{u}\cdot\frac{1}{\mu_0}\vec{B}\otimes\vec{B}$$

$$\Rightarrow \boxed{\frac{\partial H}{\partial t} + \nabla\cdot\vec{U} = 0}$$

4) $\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

$$\nabla\times\left(-\vec{u}\times\vec{B}\right) = -\frac{\partial\vec{B}}{\partial t}$$

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times\left(\vec{u}\times\vec{B}\right)$$

$$\nabla\times\left(\vec{A}\times\vec{B}\right) = \vec{A}\left(\nabla\cdot\vec{B}\right) - \vec{B}\left(\nabla\cdot\vec{A}\right)$$

$$= \left(\nabla\cdot\vec{B}\right)\vec{A} - \left(\nabla\cdot\vec{A}\right)\vec{B}$$

$$= \nabla\cdot\left(\vec{B}\otimes\vec{A} - \vec{A}\otimes\vec{B}\right)$$

$$\frac{\partial\vec{B}}{\partial t} = \nabla\cdot\left(\vec{B}\otimes\vec{u} - \vec{u}\otimes\vec{B}\right)$$

$$\frac{\partial\vec{B}}{\partial t} + \nabla\cdot\left(\underbrace{\vec{u}\otimes\vec{B} - \vec{B}\otimes\vec{u}}_{\text{def } \hat{Y}}\right) = 0$$

$$\boxed{\frac{\partial \vec{B}}{\partial t} + \nabla \cdot \hat{\underline{Y}} = \vec{0}}$$

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② Raspisati izraz $\vec{j} \times \vec{B}$ u idealnoj MHD preko $\parallel$ i $\perp$ komponente $\vec{B}$:

$$\vec{j} \times \vec{B} = \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} \ominus$$

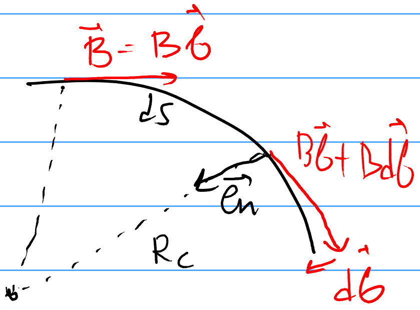
$$(\vec{A} \times \vec{B}) \times \vec{C} = \vec{C} \cdot (\vec{A} \otimes \vec{B}) - \vec{C} \cdot (\vec{B} \otimes \vec{A})$$

$$\ominus \frac{1}{\mu_0} \vec{B} (\nabla \cdot \vec{B}) - \frac{1}{\mu_0} \nabla (\vec{B} \cdot \vec{B}) =$$

$$= \frac{1}{\mu_0} \vec{B} \cdot \nabla \otimes \vec{B} - \frac{1}{\mu_0} \nabla \left(\frac{B^2}{2} \right) =$$

$$= \frac{1}{\mu_0} B\vec{b} \cdot \left(\nabla_{\parallel} + \nabla_{\perp} \right) (B\vec{b}) - \left(\nabla_{\parallel} + \nabla_{\perp} \right) \left(\frac{B^2}{2\mu_0} \right) =$$

$$\nabla_{\parallel} = \vec{b} \cdot \frac{d}{ds}$$



$$\vec{b}\cdot\vec{b} = 1 \quad ; \quad \vec{b}\cdot\vec{e}_n = 0 \quad ; \quad \nabla_{\parallel} = \vec{b}\frac{d}{ds}$$

$$(\vec{b}\cdot\nabla)\vec{b} = \frac{\vec{e}_n}{R_c} = \frac{d\vec{b}}{ds}$$

$$= \frac{1}{\mu_0} B\underbrace{\vec{b}\cdot\vec{b}}_{1}\frac{d}{ds}(B\vec{b}) + \cancel{\frac{1}{\mu_0} B\,\vec{b}\cdot\nabla_{\perp}(B\vec{b})} - \vec{b}\frac{d}{ds}\left(\frac{B^2}{2\mu_0}\right) -$$

$$\nabla_{\perp}\vec{B} \perp \vec{b}$$

$$\nabla_{\perp}\vec{B}\cdot\vec{b} = 0$$

$$- \nabla_{\perp}\left(\frac{B^2}{2\mu_0}\right) =$$

$$= \frac{1}{\mu_0} B\frac{dB}{ds}\vec{b} + \frac{1}{\mu_0} B^2 \frac{d\vec{b}}{ds} + \vec{b}\frac{d}{ds}\left(\frac{B^2}{2\mu_0}\right) - \nabla_{\perp}\left(\frac{B^2}{2\mu_0}\right)$$

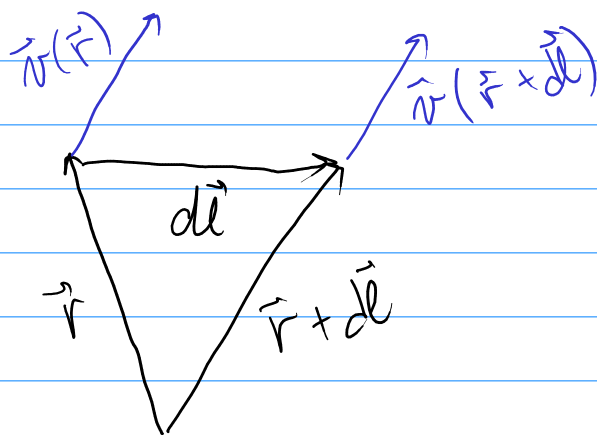
$$\frac{\vec{e}_n}{R_c}$$

$$\frac{d}{ds}\left(\frac{B^2}{2}\right) = \frac{1}{2}2B\frac{dB}{ds} = B\frac{dB}{ds}$$

$$= \cancel{\frac{1}{\mu_0}\frac{d}{ds}\left(\frac{B^2}{2}\right)\vec{b}} + \frac{1}{\mu_0} B^2 \frac{\vec{e}_n}{R_c} \cancel{- \vec{b}\frac{d}{ds}\left(\frac{B^2}{2\mu_0}\right)} - \nabla_{\perp}\left(\frac{B^2}{2\mu_0}\right)$$

$$= \boxed{\frac{1}{\mu_0} B^2 \frac{\vec{e}_n}{R_c} - \nabla_{\perp}\left(\frac{B^2}{2\mu_0}\right) = \vec{j}\times\vec{B}}$$

3) Neka se posmatra proticanje fluida u tačkama određenim $\vec{r}$ i $\vec{r}+d\vec{l}$. Odrediti jdnu koja opisuje kretanje linijskog elementa $d\vec{l}$ koji spaja te 2 tačke.



$$\frac{d}{dt}(d\vec{l}) = \frac{d}{dt}(\vec{r}+d\vec{l}) - \frac{d}{dt}(\vec{r}) =$$

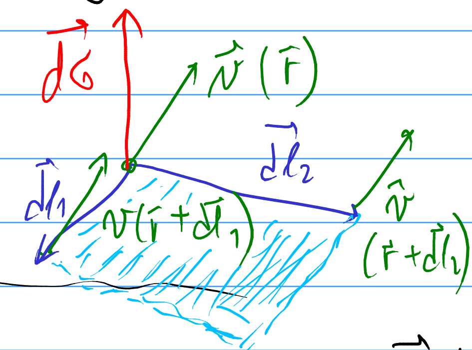
$$= \underbrace{\vec{v}(\vec{r}+d\vec{l}) - \vec{v}(\vec{r})}_{d\vec{v} \text{ pri prost. promeni } d\vec{l}} = \boxed{d\vec{l}\cdot(\nabla\otimes\vec{v})}$$

$$d\vec{A} = d\vec{r}\cdot(\nabla\otimes\vec{A})$$

$$d\vec{v} = d\vec{l}\cdot(\nabla\otimes\vec{v})$$

(4) Neka se posmatra površinski element def. kao vekt. proizvod 2 linijska elem.

$$d\vec{\sigma} = d\vec{l}_1 \times d\vec{l}_2$$



Odrediti $\frac{d}{dt}(d\vec{\sigma})$.

$$\frac{d}{dt}(d\vec{\sigma}) = \frac{d}{dt}(d\vec{l}_1 \times d\vec{l}_2) = \frac{d(d\vec{l}_1)}{dt} \times d\vec{l}_2 + d\vec{l}_1 \times \frac{d(d\vec{l}_2)}{dt}$$

$$= \left(d\vec{l}_1 \cdot (\nabla \otimes \vec{v})\right) \times d\vec{l}_2 + d\vec{l}_1 \times \left(d\vec{l}_2 \cdot (\nabla \otimes \vec{v})\right) =$$

$$(\vec{A} \times \vec{B}) \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{A}(\vec{B} \cdot \vec{C})$$

$$\vec{C} \equiv \nabla \Rightarrow \quad (\vec{B} \otimes \vec{A}) \cdot \nabla - (\vec{A} \otimes \vec{B}) \cdot \nabla$$

operator $(\vec{A} \times \vec{B}) \times \nabla$ deluje na $\vec{D}$:

$$\left[(\vec{A} \times \vec{B}) \times \nabla\right] \times \vec{D} = \left((\vec{B} \otimes \vec{A}) \cdot \nabla\right) \times \vec{D} - \left((\vec{A} \otimes \vec{B}) \cdot \nabla\right) \times \vec{D} =$$

$$= -(\vec{A} \cdot \nabla)\vec{D} \times \vec{B} + (\vec{B} \cdot \nabla)\vec{D} \times \vec{A}$$

$$\left(\left(\vec{B}\otimes\vec{A}\right)\cdot\nabla\right)\times\vec{D} = \left(\vec{B}\left(\vec{A}\cdot\nabla\right)\right)\times\vec{D} =$$

$$= \left(\vec{A}\cdot\nabla\right)\vec{B}\times\vec{D} = -\left(\vec{A}\cdot\nabla\right)\vec{D}\times\vec{B}$$

$$\vec{dl}_1 \equiv \vec{A} \quad , \quad \vec{dl}_2 \equiv \vec{B} \quad , \quad \vec{D} \equiv \vec{v}$$

$$\left[\left(\vec{dl}_1\times\vec{dl}_2\right)\times\nabla\right]\times\vec{v} = -\left(\vec{dl}_1\cdot\nabla\right)\vec{v}\times\vec{dl}_2 +$$

$$+\left(\vec{dl}_2\cdot\nabla\right)\vec{v}\times\vec{dl}_1 =$$

$$= -\left(\vec{dl}_1\cdot\nabla\right)\vec{v}\times\vec{dl}_2 - \vec{dl}_1\times\left(\vec{dl}_2\cdot\nabla\right)\vec{v} =$$

$$= -\vec{dl}_1\cdot\left(\nabla\otimes\vec{v}\right)\times\vec{dl}_2 - \vec{dl}_1\times\left(\vec{dl}_2\cdot\left(\nabla\otimes\vec{v}\right)\right)$$

$$\Leftrightarrow -\left[\left(\vec{dl}_1\times\vec{dl}_2\right)\times\nabla\right]\times\vec{v} = -\left(d\vec{\sigma}\times\nabla\right)\times\vec{v} \Leftrightarrow$$

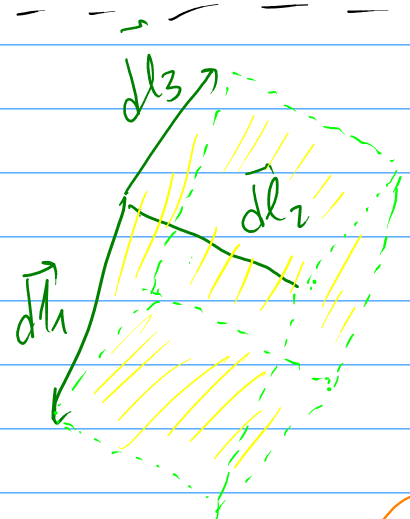
$$\left(\vec{A}\times\nabla\right)\times\vec{B} = \nabla\left(\vec{A}\cdot\vec{B}\right) - \vec{A}\left(\nabla\cdot\vec{B}\right) =$$

$$= \vec{A}\cdot\left(\nabla\otimes\vec{B}\right) - \vec{A}\left(\nabla\cdot\vec{B}\right)$$

$$\Leftrightarrow \boxed{d\vec{\sigma}\left(\nabla\cdot\vec{v}\right) - d\vec{\sigma}\cdot\left(\nabla\otimes\vec{v}\right) = \frac{d}{dt}\left(d\vec{\sigma}\right)}$$

(5) Razmotriti kinematiku pokretnog zapreminskog elementa fluida definisanog preko:

$$dV = d\vec{\sigma} \cdot d\vec{l}_3 = (d\vec{l}_1 \times d\vec{l}_2) \cdot d\vec{l}_3$$



$$\frac{d}{dt}(dV) = \frac{d}{dt}(d\vec{\sigma} \cdot d\vec{l}_3) =$$

$$= \frac{d}{dt}(d\vec{\sigma}) \cdot d\vec{l}_3 + d\vec{\sigma} \cdot \frac{d}{dt}(d\vec{l}_3) =$$

$$= \left(d\vec{\sigma}(\nabla \cdot \vec{v})\right) \cdot d\vec{l}_3 - \cancel{\left(d\vec{\sigma} \cdot (\nabla \otimes \vec{v})\right) \cdot d\vec{l}_3} +$$

$$+ d\vec{\sigma} \cdot \cancel{\left(d\vec{l}_3 \cdot (\nabla \otimes \vec{v})\right)} =$$

$$= (\nabla \cdot \vec{v})\, d\vec{l}_3 \cdot d\vec{\sigma} = (\nabla \cdot \vec{v})\, dV$$

$$\boxed{\frac{d}{dt}(dV) = (\nabla \cdot \vec{v})\, dV}$$

6. Neka se posmatra kretanje elementa fluida mase $dm = \rho dV$ u modelu idealne MHD. Dokazati da važi:

$$\frac{d}{dt}(dm) = 0$$

$$\frac{d}{dt}(dm) = \frac{d}{dt}(\rho dV) = \frac{d\rho}{dt} dV + \rho \frac{d}{dt}(dV) =$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \rho = 0$$

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{v}$$

$$= -\rho (\nabla \cdot \vec{v}) dV + \rho (\nabla \cdot \vec{v}) dV = 0$$

7 Posmatra se pokretni element fluida. Stopa promene impulsa $\vec{\pi}\,dV$ je tada $\frac{d}{dt}(\vec{\pi}\,dV) \neq 0$. Dokazati da to važi u idealnoj MHD:

$$\frac{d}{dt}(\vec{\pi}\,dV) = \frac{d\vec{\pi}}{dt}dV + \vec{\pi}\frac{d}{dt}(dV) \ominus$$

$$\boxed{\begin{gathered}\frac{\partial\vec{\pi}}{\partial t} + \nabla\cdot\hat{T} = 0 \\ \underbrace{\frac{\partial\vec{\pi}}{\partial t} + \vec{v}\cdot(\nabla\otimes\vec{\pi})}_{\frac{d\vec{\pi}}{dt}} = -\nabla\cdot\hat{T} + \vec{v}\cdot(\nabla\otimes\vec{\pi})\end{gathered}}$$

$$\ominus\ -\nabla\cdot\hat{T}\,dV + \underbrace{\vec{v}\cdot(\nabla\otimes\vec{\pi})}_{(\vec{v}\cdot\nabla)\vec{\pi}}dV + \vec{\pi}(\nabla\cdot\vec{v})\,dV \ominus$$

$$\boxed{\nabla\cdot(\vec{v}\otimes\vec{\pi}) = (\nabla\cdot\vec{v})\vec{\pi} + (\vec{v}\cdot\nabla)\vec{\pi}}$$

$$\ominus -\nabla\cdot\left(\rho\vec{v}\otimes\vec{v}+\left(p+\frac{B^2}{2\mu_0}\right)\hat{I}-\frac{1}{\mu_0}\vec{B}\otimes\vec{B}\right)dV+$$

$$\nabla\cdot\left(\vec{v}\otimes\vec{\pi}\right)dV=$$

$$=-\nabla\cdot\left(\cancel{\rho\vec{v}\otimes\vec{v}}+\left(p+\frac{B^2}{2\mu_0}\right)\hat{I}-\frac{1}{\mu_0}\vec{B}\otimes\vec{B}-\cancel{\vec{v}\otimes\rho\vec{v}}\right)dV$$

$$=\left(-\nabla\left(p+\frac{B^2}{2\mu_0}\right)+\nabla\cdot\left(\frac{1}{\mu_0}\vec{B}\otimes\vec{B}\right)\right)dV$$

$$=\left(-\nabla p\underbrace{-\nabla\left(\frac{B^2}{2\mu_0}\right)+\nabla\cdot\left(\frac{1}{\mu_0}\vec{B}\otimes\vec{B}\right)}_{+\vec{j}\times\vec{B}\ (1.\ \text{zadatak})}\right)dV$$

$$\frac{d}{dt}\left(\vec{\pi}\,dV\right)=\left(-\nabla p+\vec{j}\times\vec{B}\right)dV\neq 0$$

u opštem slučaju!