ЈУН2:

$\boxed{1}$ а) $\{A_n\}_{n\in\mathbb{N}}$ $\quad ?\ \sup\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sup\{\sup A_n \mid n\in\mathbb{N}\}$?

$*$: Нека је $S = \sup\{\sup A_n \mid n\in\mathbb{N}\}$

$\Rightarrow \forall n\in\mathbb{N}\ \ \sup A_n \le S \quad$ и $\quad \forall\varepsilon>0\ \exists n\in\mathbb{N}\ \ S < \sup A_n - \varepsilon/2$

$\underbrace{\forall n\in\mathbb{N}\ \forall a\in A_n\ \ a \le \sup A_n}$ и $\underbrace{\forall n\in\mathbb{N}\ \forall\varepsilon>0\ \exists a\in A_n\ \ \sup A_n < a - \varepsilon/2}$

$\Rightarrow \forall n\in\mathbb{N}\ \forall a\in A_n\ \ a\le S \quad$ и $\quad \forall\varepsilon>0\ \exists n\in\mathbb{N}\ \exists a\in A_n\ \ S < a-\varepsilon$

$\Rightarrow \forall a\in\bigcup_{n\in\mathbb{N}} A_n\ \ a\le S \quad$ и $\quad \forall\varepsilon>0\ \exists a\in\bigcup_{n\in\mathbb{N}} A_n\ \ S<a-\varepsilon$

$\Rightarrow S = \sup\left(\bigcup_{n\in\mathbb{N}} A_n\right)$.

б) $A_n = \{\sqrt[n]{m+n} - \sqrt[n]{m-n} \mid m\ge n\}$, $n\in\mathbb{N}$; $\sup A_n, \inf A_n, \sup\bigcup_{n\in\mathbb{N}} A_n = ?$

— $n\in\mathbb{N}$ фиксирано

$f_n(x) = \sqrt[n]{x+n} - \sqrt[n]{x-n} = (x+n)^{1/n} - (x-n)^{1/n}$, $x\ge n$

$f_n'(x) = \frac{1}{n}(x+n)^{1/n-1} - \frac{1}{n}(x-n)^{1/n-1}$

$x+n > x-n$, $\frac{1}{n}-1 \le 0 \Rightarrow f_n' \le 0 \Rightarrow f_n \downarrow \quad x\ge n$

$\Rightarrow \sup A_n = \max A_n = f_n(n) = \sqrt[n]{2n}$

$\inf A_n = \lim_{x\to+\infty} f_n(x) = \lim_{x\to+\infty} (x+n)^{1/n} - (x-n)^{1/n} = \lim_{x\to+\infty} x^{1/n}\left(\left(1+\frac{n}{x}\right)^{1/n} - \left(1-\frac{n}{x}\right)^{1/n}\right)$ $\quad \frac{n}{x}\to 0$

$= \lim_{x\to+\infty} x^{1/n}\left(\cancel{1} + \frac{1}{n}\cdot\frac{n}{x} + o\left(\frac{1}{x}\right) - \left(\cancel{1} - \frac{1}{n}\cdot\frac{n}{x} + o\left(\frac{1}{x}\right)\right)\right)$

$= \lim_{x\to+\infty} \frac{2}{x^{1-1/n}} + o\left(\frac{1}{x^{1-1/n}}\right) = \begin{cases} 0, & n>1 \\ 2, & n=1 \end{cases}$

за $n>1$ $\quad \inf A_n = 0$, min се не достиже

$n=1$ $\quad \inf A_n = \min A_n = 2$.

$\sup\left(\bigcup_{n\in\mathbb{N}} A_n\right) \overset{а)}{=} \sup\{\sup A_n \mid n\in\mathbb{N}\} = \sup\{\underbrace{(2n)^{1/n}}_{= e^{1/n\cdot\ln 2n}} \mid n\in\mathbb{N}\}$

$x\ge 1$: $f(x) = e^{\frac{1}{x}\ln 2x}$, $f'(x) = e^{1/x\ln 2x}\cdot\left(\frac{1}{x^2} - \frac{\ln 2x}{x^2}\right) = e^{1/x\ln 2x}\,\frac{1-\ln 2x}{x^2}$

знак f' зависи од $\underbrace{1-\ln 2x}_{g(x)}$ за $x\ge 1$, $g'(x) = -\frac{1}{x} < 0$, $g\downarrow$, $g(1) = 1>0$, $g(2)<0$

дакле $f'<0$ за $x\ge 2$ $\Rightarrow f\downarrow$ за $x\ge 2$ $\quad f(2) = 2$, $f(1) = 2$

$\Rightarrow \sup\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 2 = \max\left(\bigcup_{n\in\mathbb{N}} A_n\right)$.

$\boxed{2}$ $f: (0,+\infty)\to\mathbb{R}$, $f(t) = \begin{cases} \frac{\cos^2 t}{t}, & t\ge\frac{\pi}{2} \\ 0, & 0<t<\frac{\pi}{2} \end{cases}$. $\quad F: (0,+\infty)\to\mathbb{R}$, $F(x) = \int_0^x f(t)\,dt$

F равн непр на $(0,+\infty)$?

— f непр? $\lim_{t\to\frac{\pi}{2}-} f(t) = 0 = f\left(\frac{\pi}{2}\right) = \lim_{t\to\frac{\pi}{2}+} f(t) \Rightarrow f$ непр на $(0,+\infty)$

$\Rightarrow F$ диф на $(0,+\infty)$ и $F'(x) = f(x)$

$x\ge\frac{\pi}{2}$: $|f(x)| = \frac{\cos^2 x}{x} \le \frac{1}{x} \le \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

$x\in\left(0,\frac{\pi}{2}\right)$: $|f(x)| = 0 \le \frac{2}{\pi}$

$|F(x) - F(x_0)| \underset{\text{Лагранж},\ \xi\in(x,x_0)}{=} |F'(\xi)|\cdot|x-x_0| = |f(\xi)|\cdot|x-x_0| \le \frac{2}{\pi}|x-x_0|$

$\Rightarrow F$ Липшицова на $(0,+\infty)$

$\Rightarrow F$ равн непр на $(0,+\infty)$.

$\boxed{3}$ $I(a) = \int\limits_{-\infty}^{+\infty} \frac{\operatorname{arctg} e^x}{e^{ax}+e^{-ax}}\,dx = ?$, конв ?

Hint : $\operatorname{arctg} t + \operatorname{arctg}\frac{1}{t} = \frac{\pi}{2}$ ⊛ , $t>0$

$h(t) = \operatorname{arctg} t + \operatorname{arctg}\frac{1}{t} \Rightarrow h'(t) = \frac{1}{1+t^2} + \frac{1}{1+\frac{1}{t^2}}\cdot\left(-\frac{1}{t^2}\right) = 0 \Rightarrow h = const$

$h(1) = \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}$

$\Rightarrow h(t) = \frac{\pi}{2}$, $t>0$.

$a \neq 0$:

$$I(a) = \int\limits_{-\infty}^{+\infty} \frac{\operatorname{arctg} e^x}{e^{ax}+e^{-ax}}\,dx = \int\limits_{0}^{+\infty} \frac{\operatorname{arctg} e^x}{e^{ax}+e^{-ax}}\,dx + \underbrace{\int\limits_{-\infty}^{0} \frac{\operatorname{arctg} e^x}{e^{ax}+e^{-ax}}\,dx}_{\text{смена } y=-x,\ dy=-dx} =$$

$$= \int\limits_{0}^{+\infty} \frac{\operatorname{arctg} e^x}{e^{ax}+e^{-ax}}\,dx + \underbrace{\int\limits_{0}^{+\infty} \frac{\operatorname{arctg} e^{-y}}{e^{-ay}+e^{ay}}\,dy}_{y=x} = \left[e^{-x} = \frac{1}{e^x} \right]$$

$$= \int\limits_{0}^{+\infty} \frac{\operatorname{arctg} e^x + \operatorname{arctg}\frac{1}{e^x}}{e^{ax}+e^{-ax}}\,dx \underset{t=e^x}{\overset{⊛}{=}} \int\limits_{0}^{+\infty} \frac{\frac{\pi}{2}}{e^{ax}+e^{-ax}}\,dx =$$

$$= \frac{\pi}{2}\int\limits_{0}^{+\infty} \frac{e^{ax}\,dx}{e^{2ax}+1} = \left[\begin{array}{l} u = e^{ax} \\ du = a\cdot e^{ax}\,dx \end{array} \right]$$

→ у зависности од sgn a имамо другачије границе новог интеграла

$$\underset{1^\circ\ a>0}{\overset{\odot}{=}} \frac{\pi}{2a}\int\limits_{1}^{+\infty} \frac{du}{u^2+1} = \frac{\pi}{2a}\cdot \operatorname{arctg} u \Big|_1^{+\infty} = \frac{\pi^2}{8a}$$

$$\underset{2^\circ\ a<0}{\overset{\odot}{=}} \frac{\pi}{2a}\int\limits_{1}^{0} \frac{du}{u^2+1} = \frac{\pi}{2a}\cdot \operatorname{arctg} u \Big|_1^{0} = -\frac{\pi^2}{8a}$$

$\Rightarrow a\neq 0$ $I(a)$ конв и $I(a) = \frac{\pi^2}{8|a|}$

$a=0$: $I(0) = \int\limits_{-\infty}^{+\infty} \frac{\operatorname{arctg} e^x}{2}\,dx$, $\operatorname{arctg} e^x \to \frac{\pi}{2}$, $x\to+\infty$

$\Rightarrow$ $I(0)$ дивергира !

$\boxed{4}$ a) $a_{n+1} = \frac{a_n^2}{\sqrt{e^{a_n^2}-1}}$, $a_1 = 1$ конв ? $\lim\limits_{n\to+\infty} a_n = ?$

— $a_{n+1} = \frac{(a_n^2)^{\geq 0}}{(\sqrt{e^{a_n^2}-1})^{\geq 0}} \geq 0$ ако је низ добро деф, тј. ако $\sqrt{e^{a_n^2}-1} \neq 0$, тј. $a_n \neq 0$.

БИ: $a_1 = 1 > 0$.

ИК $a_n > 0 \Rightarrow e^{a_n^2}-1>0 \Rightarrow a_{n+1}>0$.

$\Rightarrow a_n > 0$ $\forall n\in\mathbb{N}$

монотоност?

$$a_{n+1} \boxed{?} a_n$$

$$\frac{a_n^2}{\sqrt{e^{a_n^2}-1}} \ \square \ a_n \qquad / \cdot \frac{\sqrt{e^{a_n^2}-1}}{a_n}, \quad a_n>0$$

$$a_n \ \square \ \sqrt{e^{a_n^2}-1} \qquad /^2$$

$$a_n^2 \ \square \ e^{a_n^2}-1$$

☺ $\left[g(t) = e^t - 1 - t, \quad g'(t) = e^t - 1 > 0, \quad t>0 \right.$
$\left. \Rightarrow g \uparrow \text{ и } g(0)=0 \Rightarrow g(t)>0, \quad t>0 \right]$

$\Rightarrow t < e^t - 1, \quad t>0$

$\Rightarrow a_n^2 < e^{a_n^2}-1 \Rightarrow a_{n+1} < a_n \Rightarrow a_n \downarrow$ и $a_n>0$
$\overset{\text{ТМК}}{\Rightarrow}$ a_n конвергира и $\lim\limits_{n\to+\infty} a_n \geq 0$.

$$a = \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} a_{n+1} = \lim_{n\to+\infty} \frac{a_n^2}{\sqrt{e^{a_n^2}-1}} = \frac{a^2}{\sqrt{e^{a^2}-1}}$$

$\Rightarrow a^2 = e^{a^2} - 1$ из ☺ $\Rightarrow \underline{a=0}$.

б) $a_n^2 \sim \frac{c}{n}, \quad n\to+\infty$; $c=?$

$$\Rightarrow 1 = \lim_{n\to+\infty} \frac{a_n^2}{\frac{c}{n}} = \frac{1}{c} \lim_{n\to+\infty} a_n^2 \cdot n = \frac{1}{c} \cdot \lim_{n\to+\infty} \frac{n}{\frac{1}{a_n^2}} \overset{\text{ШТ}}{=} \left(\frac{1}{a_n^2} \to +\infty\right)$$

$$= \frac{1}{c} \lim_{n\to+\infty} \frac{1}{\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}} = \frac{1}{c} \lim_{n\to+\infty} \frac{1}{\frac{e^{a_n^2}-1}{a_n^4} - \frac{1}{a_n^2}} = \frac{1}{c} \lim_{n\to+\infty} \frac{a_n^4}{e^{a_n^2}-1-a_n^2} =$$

$\left[e^t \sim 1+t+\frac{t^2}{2}, \ t\to 0 \right]$

$$= \frac{1}{c} \lim_{n\to+\infty} \frac{a_n^4}{\cancel{1}+\cancel{a_n^2} + \frac{a_n^4}{2} + o(a_n^4) - \cancel{1} - \cancel{a_n^2}} = \frac{1}{c} \lim_{n\to+\infty} \frac{1}{\frac{1}{2} + o(1)} = \frac{1}{c} \cdot 2$$

$\Rightarrow \boxed{c=2} \Rightarrow a_n \sim \frac{\sqrt{c}}{\sqrt{n}}, \ n\to+\infty \Rightarrow a_n \sim \frac{\sqrt{2}}{\sqrt{n}}, \ n\to+\infty$.

в) $\sum\limits_{n=1}^{\infty} (-1)^n \operatorname{arctg} \frac{1}{\sqrt{n}} \cdot \sin a_n$ конв? $\quad \sin t \sim t, \ t\to 0$; $\operatorname{arctg} t \sim t, \ t\to 0$

$0<a_n<1 \Rightarrow 0 < \sin a_n < \sin 1$

$$\Rightarrow \left|(-1)^n \operatorname{arctg} \frac{1}{\sqrt{n}} \cdot \sin a_n\right| = \operatorname{arctg} \underbrace{\frac{1}{\sqrt{n}}}_{\to 0} \cdot \sin \underbrace{a_n}_{\to 0} \sim \frac{1}{\sqrt{n}} \cdot a_n \overset{б)}{\sim} \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{2}}{\sqrt{n}} \sim \frac{\sqrt{2}}{n}, \ n\to+\infty$$

$\sum \frac{1}{n}$ див. $\underset{\text{II п.к.}}{\Rightarrow} \sum\limits_{n=1}^{\infty} \left|(-1)^n \operatorname{arctg} \frac{1}{\sqrt{n}} \sin a_n\right|$ дивергира.

$a_n \downarrow$ и $a_n \to 0 \Rightarrow \sin a_n \downarrow$ и $\sin a_n \to 0$
$\operatorname{arctg} \frac{1}{\sqrt{n}} \downarrow$

$\Rightarrow \lim\limits_{n\to+\infty} \operatorname{arctg} \frac{1}{\sqrt{n}} \cdot \sin a_n = 0$ и $\operatorname{arctg} \frac{1}{\sqrt{n}} \cdot \sin a_n \downarrow \overset{\text{Лајбниц}}{\Rightarrow} \sum (-1)^n \operatorname{arctg} \frac{1}{\sqrt{n}} \sin a_n$ конвергира