илићев ...

II КОЛОКВИЈУМ РЕШЕЊА:

$\boxed{1}$ $f: \mathbb{R} \to \mathbb{R}$ непр, $f(0) \neq 0$; $\lim\limits_{x\to+\infty} f(x) = 1$, $\alpha \in \mathbb{R}$

$$\lim_{n\to\infty} \int_0^1 f(nx)\, x^\alpha dx = ?$$

$\int_0^1 f(nx) x^\alpha dx$ конв ?

за $\alpha < 0$, 0 је сингуларитет

$f(nx)\, x^\alpha \sim x^\alpha \cdot f(0)$, $x \to 0$

$\Rightarrow$ за $\alpha \leq -1$ акко $\int_0^1 x^\alpha$ дивергира

II п.к. $\iff$ $\int_0^1 f(nx) x^\alpha dx$ дивергира за $\alpha \leq -1$
конв. $\alpha > -1$

$$\int_0^1 f(nx)\cdot x^\alpha dx = \underbrace{\int_0^{1/2} f(nx)\cdot x^\alpha dx}_{\downarrow \atop f(0)\cdot(+\infty)} + \underbrace{\int_{1/2}^1 f(nx)\cdot x^\alpha dx}_{\in \mathbb{R}} = f(0)\cdot(+\infty)$$

$\alpha \leq -1$:

$$\lim_{n\to+\infty} \int_0^1 f(nx)\, x^\alpha dx = \operatorname{sgn}(f(0))\cdot\infty .$$

за $\alpha > -1$:

$$\int_0^1 f(nx) x^\alpha dx = \begin{bmatrix} t = nx \\ x = t/n \\ dx = \frac{dt}{n} \end{bmatrix} = \int_0^n f(t) \frac{t^\alpha}{n^\alpha}\frac{dt}{n} = \frac{\int_0^n f(t) t^\alpha dt}{n^{\alpha+1}}$$

$$\lim_{n\to+\infty} \frac{\int_0^n f(t)\cdot t^\alpha dt}{n^{\alpha+1}} = \text{[}n^{\alpha+1} \uparrow \text{ за } \alpha+1>0, \ \alpha>-1 \text{ и } \lim_{n\to+\infty} n^{\alpha+1} = +\infty \text{]} \underset{=}{\text{ШТ}}$$

$$= \lim_{n\to+\infty} \frac{\int_0^{n+1} f(t)\, t^\alpha dt - \int_0^n f(t) t^\alpha dt}{(n+1)^{\alpha+1} - n^{\alpha+1}} = \lim_{n\to\infty} \frac{\int_n^{n+1} f(t)\cdot t^\alpha dt}{(n+1)^{\alpha+1} - n^{\alpha+1}} =$$

$\underset{\xi_n \in [n, n+1]}{\overset{\text{Тоср.вр.}}{=}} \lim\limits_{n\to\infty} f(\xi_n) \dfrac{\int_n^{n+1} t^\alpha dt}{(n+1)^{\alpha+1} - n^{\alpha}}$

$$= \lim_{n\to\infty} \frac{f(\xi_n)\left(\frac{(n+1)^{\alpha+1}}{\alpha+1} - \frac{n^{\alpha+1}}{\alpha+1}\right)}{(n+1)^{\alpha+1} - n^{\alpha+1}} = \frac{\lim\limits_{n\to+\infty} f(\xi_n)}{\alpha+1}$$

$$= \frac{1}{\alpha+1} \quad \text{за } \alpha > -1 .$$

$\xi_n \in (n, n+1)$

~~Т.о.(2)~~ $\Rightarrow f(\xi_n) \to 1,\ n\to+\infty$

Хајне

$\boxed{2}$ $\displaystyle\int_0^{2022} \frac{\sqrt[3]{x}\,\operatorname{arctg} x^{3/2}}{\ln(1+x^2)\sin\sqrt{x}}\,dx$ конв? $\quad \ln(1+x^2)\sin\sqrt{x} = 0?$

$x = ?$

$\ln(1+x^2) = 0 \Leftrightarrow x^2 = 0$

$\sin\sqrt{x} = 0 \Leftrightarrow \sqrt{x} = k\pi$

$x = 0$ – синг.

и $x = k^2\pi^2$ синг. за $k^2\pi^2 \in (0, 2022]$

$$\underbrace{\int_0^{2022} \frac{\sqrt[3]{x}\,\operatorname{arctg} x^{3/2}}{\ln(1+x^2)\sin\sqrt{x}}\,dx}_{\text{ово конв. акко конв. сваки з десне стране}} := \int_0^{\pi^2} \cdots dx + \int_{\pi^2}^{4\pi^2} \cdots + \cdots$$

$\displaystyle\int_0^{\pi^2} \frac{\sqrt[3]{x}\,\operatorname{arctg} x^{3/2}}{\underbrace{\ln(1+x^2)\sin\sqrt{x}}_{\geq 0}}\,dx$ конв? $\quad \displaystyle\int_0^{\pi^2} = \int_0^1 + \int_1^{\pi^2}$

и 0 и π^2 синг.

у 0: $\displaystyle \frac{\sqrt[3]{x}\cdot\operatorname{arctg} x^{3/2}}{\ln(1+x^2)\sin\sqrt{x}} \sim \frac{x^{1/3}\cdot x^{3/2}}{x^2\cdot x^{1/2}} \sim \frac{1}{x^{2/3}},\ x\to 0$

$\displaystyle\int_0^1 \frac{1}{x^{2/3}}\,dx$ конв.

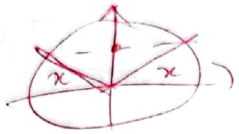
у π^2 можемо да уведемо смену $\pi^2 - t = x$

$$\int_1^{\pi^2} \frac{\sqrt[3]{x}\,\operatorname{arctg} x^{3/2}}{\ln(1+x^2)\sin\sqrt{x}}\,dx = \int_0^{\pi^2-1} \underbrace{\frac{\sqrt[3]{\pi^2-t}\ \operatorname{arctg}(\pi^2-t)^{3/2}}{\ln(1+(\pi^2-t)^2)\ \sin(\sqrt{\pi^2-t})}}_{g(t)}\,dt$$

сада је 0 сингуларитет

$$g(t) \sim \frac{1}{\sin(\sqrt{\pi^2 - t})}, \quad t \to 0^+$$

$$\sin(\sqrt{\pi^2-t}) = \sin(\underbrace{\pi - \sqrt{\pi^2 - t}}_{\to 0}) \sim \pi - \sqrt{\pi^2 - t}$$



$$= \pi - \pi\left(1 - \frac{t}{\pi^2}\right)^{1/2}$$

$$\sim \frac{t}{2\pi^2}, \quad t \to 0$$

$$g(t) \sim \frac{2\pi^2}{t}, \quad t \to 0 \qquad \int_0^{\pi^2 - t} \frac{2\pi^2}{t}\,dt \text{ див.}$$

II п.к.

$$\Rightarrow \int_0^{\pi^2-1} g(t)\,dt \text{ див.} \Rightarrow \int_A^{\pi^2} \frac{\sqrt[3]{x}\,\operatorname{arctg} x^{3/2}}{\ln(1+x^2)\sin\sqrt{x}}\,dx \text{ див.}$$

$$\Rightarrow \int_0^{\pi^2} \dots dx \text{ див.} \Rightarrow \int_0^{2022} \dots dx \text{ див.}$$

$\boxed{3}$ а)

$$I_n = -2\int_{e^{\pi/4}}^{e^{(2n)!\pi}} \sin(\ln x)\,dx = \left[\begin{array}{l}\text{смена } t = \ln x \\ x = e^t \\ dx = e^t dt\end{array}\right] =$$

$$= -2\int_{\pi/4}^{(2n)!\pi} e^t \sin t\,dt$$

$$J = \int e^t \sin t\,dt = \left[\begin{array}{l} u = \sin t \to du = \cos t\,dt \\ dv = e^t dt \to v = e^t \end{array}\right] =$$

$$= e^t \sin t - \int e^t \cos t\,dt = \left[\begin{array}{l} u = \cos t \to du = -\sin t\,dt \\ dv = e^t dt \to v = e^t \end{array}\right] \to$$

$$= e^t \sin t - e^t \cos t \cancel{+} \underbrace{\int e^t \sin t\,dt}_{J}$$

$$\Rightarrow J = \frac{1}{2}(e^t \sin t - e^t \cos t)$$

$$I_n = -2 \cdot \frac{1}{2} \cdot e^t(\sin t - \cos t)\Big|_{\pi/4}^{(2n)!\pi} =$$

$$= -e^{(2n)!\pi}(\underbrace{\sin(2n)!\pi}_{=0} - \underbrace{\cos(2n)!\pi}_{=1}) + e^{\pi/4}\cdot(\underbrace{\sin\pi/4 - \cos\pi/4}_{=0})$$

$$= e^{(2n)!\pi}$$

б) $a_n = \frac{(n+1)\pi}{\ln I_n} = \frac{n+1}{(2n)!}$

$D = ? \quad \sum_{n=1}^{\infty} a_n x^n$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+2}{(2n+2)!}}{\frac{n+1}{(2n)!}} = \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \frac{1}{(2n+2)\cdot(2n+1)} = 0$$

$\Rightarrow R = +\infty \quad \Rightarrow D = \mathbb{R}$

в) $\sum_{n=1}^{\infty} a_n \cdot (-1)^n = ?$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (n+1)}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{(2n)!} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$$

оба реда
здесне стр.
конв.

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} =$$

$$\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

$x = 1$ у нашем случају

$$= -\frac{1}{2} \sin 1 + \cos 1 - 1.$$

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$\boxed{4}$ $$\sum_{n=1}^{\infty}\underbrace{\left(\frac{(2n-1)!!}{(2n)!!}\right)^p \frac{1}{4^{qn}}}_{a_n} \quad \text{конв ?}$$

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{(2n+1)!!}{(2n+2)!!}\right)^p}{\left(\frac{(2n-1)!!}{(2n)!!}\right)^p}\cdot\frac{1/4^{q(n+1)}}{1/4^{qn}} =$$

$$= \left(\underset{\underset{1}{\downarrow\, n\to+\infty}}{\frac{2n+1}{2n+2}}\right)^p \cdot \frac{1}{4^q} \longrightarrow \frac{1}{4^q}, \quad n\to+\infty$$

Даламбер:

$\boxed{q>0}$ $\frac{1}{4^q} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ $\boxed{\text{конв}}$

$\boxed{q<0}$ $\frac{1}{4^q} > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ $\boxed{\text{див.}}$

$\boxed{q=0}$: $$\frac{a_{n+1}}{a_n} = \left(\frac{2n+1}{2n+2}\right)^p = \left(1-\frac{1}{2n+2}\right)^p$$

$$\sim 1 - \frac{p}{2n+2}$$

или ако хоћемо Гаусов крит:

$$\frac{a_n}{a_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \left(1+\frac{1}{2n+1}\right)^p$$

$$\sim 1 + \frac{p}{2n+1} \sim 1 + \frac{p/2}{n}, \quad n\to+\infty$$

Гаус: $\sum a_n$ конв. акко $p/2 > 1$

$\boxed{p>2}$ конв.

$\boxed{p\le 2}$ див.