Real submanifolds of codimension 2 of a complex space form

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Let M be a real submanifold of a complex manifold $\overline{M}$ and J be the natural almost complex structure of $\overline{M}$. For $x \in M$, we call the subspace $H_x(M) = JT_x(M) \cap T_x(M)$ of the tangent space $T_x(M)$ the holomorphic tangent space of M. If the holomorphic tangent space has constant dimension with respect to $x \in M$, the submanifold is called a CR submanifold and the constant complex dimension is called the CR dimension of M.
It is well known that an n-dimensional real hypersurface of a complex manifold is a CR submanifold of CR dimension $\frac{n-1}{2}$.
We consider now real submanifolds of codimension 2 of a complex manifold. Then contrary to real hypersurfaces, the submanifolds are something complicated. They are not only CR submanifolds of CR dimension $\frac{n-2}{2}$, but also some other cases. For example a complex hypersurface is a real submanifold of codimension 2. Moreover there exists a submanifold which is not CR submanifold. However, to investigate even dimensional real submanifolds of complex manifold, codimension 2 case is fundamental.
We investigate real submanifolds of codimension 2 of a complex manifold under the condition that

$$h(FX, Y) + h(X, FY) = 0 \qquad (*)$$

and obtained the following results.

Theorem 1. If a complex hypersurface M of a Kähler maniold $\overline{M}$ satisfies the condition (*), M is a totally geodesic submanifold.

Theorem 2. Let $\overline{M}$ be a non Euclidean complex space form. If a real submanifold M of codimension 2 satisfies the condition (*), then one of the following holds.
(1) M is a totally geodesic complex hypersurface.
(2) M is a CR submanifold of CR dimension $\frac{n-2}{2}$ with $\lambda = 0$.

Theorem 3. Let M be a real submanifold of codimension 2 of coplex Euclidean space $\mathbf{C}^{\frac{n+2}{2}}$ which satisfies the condition (*). Then M is one of the following:
(1) M is an n-dimensional Euclidean space $\mathbf{E}^n$,
(2) M is an n-dimensional sphere $\mathbf{S}^n$,
(3) M is a product of an even dimensinal sphere with Euclidean sspace $\mathbf{S}^r \times \mathbf{E}^{n-r}$
(4) M is a CR submanifold of CR dimension $\frac{n-2}{2}$ with $\lambda = 0$.
Where λ is a function defined on the submanifold M.