

To the Teaching-Scientific Council  
of the Faculty of Mathematics,  
University of Belgrade

At the Teaching-Scientific Council meeting at the Faculty of Mathematics of the University of Belgrade on 24.09.2021 we have been appointed members of the committee for evaluation of the doctoral dissertation under the name:

*‘Semi-Fredholm operators on Hilbert  $C^*$ -modules’*

written by the candidate Stefan Ivković. After reading the manuscript submitted by Stefan Ivković we submit to the Teaching-Scientific Council of the Faculty of Mathematics the following

## **REPORT**

### **1. Candidate's CV**

Stefan Ivković was born on 03.08.1989 in Jagodina, Serbia. At the age of 12, he moved together with his parents to Norway. He has studied mathematics at the Faculty of Mathematics and Natural Sciences, University of Oslo, where he obtained master's degree in mathematics in 2016. After completing his education at University of Oslo, he has been accepted to PhD-studies in mathematics at the Faculty of Mathematics, University of Belgrade. He was ranked as no. 5, with average grade 9,89 (in the Serbian system of grading) achieved on his studies at University of Oslo. Until now he has taken all mandatory 8 exams at PhD-studies with the maximal grade 10. In addition, he has published 7 papers in SCI journals, in 5 of which he was a single author. He has given talks at 4 international conferences as an invited speaker. He has also presented results from his thesis at the seminars at Lomonosov Moscow State University, at Department of Mathematics and Informatics of University of Palermo and at Mathematical Institute SANU in Belgrade. He has been engaged as a referee for Banach Journal of Mathematical Analysis and for Linear and Multilinear Algebra. From May 2018 he has been employed at the Mathematical Institute SANU in Belgrade where he works now as a research assistant. In addition to his education in mathematics, he has also obtained a master's degree from Norwegian Academy of Music as a classical solo performing pianist.

### **2. Candidate's scientific work (publications and talks at the conferences)**

#### **Publications:**

1. Ivković, S. *Semi-Fredholm theory on Hilbert  $C^*$ -modules*, Banach J. Math. Anal., **13** (2019), no. 4, 989-1016.
2. Ivković, S. *On operators with closed range and semi-Fredholm operators over  $W^*$ -algebras*, Russ. J. Math. Phys. **27** (2020), 48–60
3. Ivković, S. *On compressions and generalized spectra of operators over  $C^*$ -algebras*. Ann. Funct. Anal. **11** (2020), 505–522.

4. Ivković, S. *On various generalizations of semi-A-Fredholm operators*. Complex Anal. Oper. Theory **14** (2020), 41.
5. Ivković, S. *On Upper Triangular Operator  $2 \times 2$  Matrices Over  $C^*$ -Algebras*, FILOMAT, **34** (2020), no. 3, 691-706.
6. Tabatabaie, S.M., Ivković, S. *Linear dynamics of discrete cosine functions on solid Banach function spaces*. Positivity **25** (2021), 1437–1448.
7. Ivković, S., Tabatabaie, S.M. *Hypercyclic Translation Operators on the Algebra of Compact Operators*. Iran J. Sci. Technol. Trans. Sci. **45** (2021), 1765–1775.

### Talks at conferences:

1. “Semi-Fredholm theory on Hilbert  $C^*$ -modules”. WHO 19 International Workshop on Harmonic Analysis and Operator Theory 26–29 August 2019, Istanbul University, Istanbul, Turkey
2. “The generalized spectra of operators over  $C^*$ -algebras”. HARMONIC AND SPECTRAL ANALYSIS. International Zoom Conference, Debrecen, Hungary, 8-10 June 2020
3. “Semi-Fredholm theory on Hilbert  $C^*$ -modules”. International Workshop Hilbert  $C^*$  - Modules Online Weekend in memory of William L. Paschke, Lomonosov University, Moscow, Russia, December 5-6, 2020
2. “Semi-Fredholm operators on Hilbert  $C^*$ -modules”. HARMONIC AND SPECTRAL ANALYSIS. International Zoom Conference, Debrecen, Hungary, 31 May-2 June 2021

### 3. The topic and the aim of the dissertation

Hilbert  $C^*$ -modules provide a natural generalization of Hilbert spaces arising when the field of complex numbers as scalars is replaced by an arbitrary  $C^*$ -algebra. This generalization, in the case of commutative  $C^*$ -algebras, was suggested by I. Kaplansky, and the general theory of Hilbert  $C^*$ -modules (i.e. over an arbitrary  $C^*$ -algebra serving as ‘scalars’) has appeared about 50 years ago in the pioneering paper by W. L. Paschke. This theory has proved to be a convenient tool in the theory of operator algebras, allowing to obtain information about  $C^*$ -algebras by studying Hilbert  $C^*$ -modules over them, e.g. an important notion of Morita equivalence for  $C^*$ -algebras was formulated in terms of Hilbert  $C^*$ -modules. On the other hand, Hilbert  $C^*$ -modules naturally appear in the study of (pseudo)differential operators acting on sections of vector bundles over manifolds with some additional structure. For example, operators compatible with a cocompact action of a group on a noncompact manifold can be viewed as operators in Hilbert  $C^*$ -modules over the group  $C^*$ -algebra. This approach is very useful, as many elliptic operators on noncompact manifolds are not Fredholm, while being considered as operators on Hilbert  $C^*$ -modules they turn out to be Fredholm. The general theory of Fredholm operators on Hilbert  $C^*$ -modules was developed by A. S. Mishchenko and A. T. Fomenko. Kernel and cokernel of such operators need not be free finitely generated modules any more, but they can be placed inside ones, and one still can define the analytical index and prove that it coincides with the topological index.

Semi-Fredholm operators represent an interesting class of operators: while Fredholm operators are invertible modulo compact operators, they are one-sided invertible modulo compacts. They still have an index, which now may be infinite, thus less informative, and while there is no relation with ellipticity, stability under compact perturbations makes the class of semi-Fredholm operators still interesting.

The aim of the Ivković's thesis is to develop the theory of semi-Fredholm operators on Hilbert  $C^*$ -modules. As was mentioned, kernels and cokernels of such operators on Hilbert  $C^*$ -modules need not be finite-dimensional in any reasonable way – they may be only embedded into finitely generated free submodules, so the author had to overcome serious technical difficulties on his way. Besides providing definitions of semi-Weyl and semi-Fredholm operators on Hilbert  $C^*$ -modules, the author has obtained several results generalizing the counterparts from the classical semi-Fredholm theory on Hilbert spaces. Another aspect of Ivković's research is a generalization of the notion of the spectrum for operators on Hilbert  $C^*$ -modules. The idea is to replace scalars by elements of the center of the  $C^*$ -algebra involved. On this way, generalizations of some results from the classical spectral semi-Fredholm theory are obtained, e.g. the results by J. Zemánek regarding the relationship between the spectra of an operator and the spectra of its compressions.

#### **4. The content of the dissertation**

The dissertation consists of Introduction (Chapter 1, 5 pages), Preliminaries (Chapter 2), seven chapters (Chapters 3-9), Final Remarks (Chapter 10) – totally 145 pages, followed by Bibliography with 56 items, and Index with the list of notations.

Chapter 3, Semi- $C^*$ -Fredholm operators, is the central part of the thesis and consists of 7 sections. In Section 3.1, the definition of adjointable semi-Fredholm operators on the standard Hilbert  $C^*$ -module is given and some of their basic properties are proved. In Section 3.2, the set of upper semi- $C^*$ -Fredholm operators is described in terms of some equivalent conditions, generalizing the Schechter characterization. In Section 3.3, it is shown that the set of upper semi- $C^*$ -Fredholm operators is open with respect to the norm topology. In Section 3.4, certain classes of operators on the standard Hilbert  $C^*$ -module are constructed and studied, as a generalization of upper and lower semi-Weyl operators on Hilbert spaces. In Section 3.5, the previous results are extended to the bounded operators which are not necessarily adjointable (recall that on Hilbert spaces any bounded operator is adjointable, while on Hilbert  $C^*$ -modules this is no longer true). In Section 3.6, the same is done for semi- $C^*$ -Weyl operators. Finally, Section 3.7 contains some examples illustrating the above results.

Chapter 4 deals with the case when the  $C^*$ -algebra is a  $W^*$ -algebra. Hilbert  $C^*$ -modules over  $W^*$ -algebras share more properties with Hilbert spaces than general Hilbert  $C^*$ -modules. Accordingly, semi-Fredholm operators in this case share more properties of their counterparts on Hilbert spaces. In particular, a generalization of Schechter-Lebow characterization of semi-Fredholm operators and the punctured neighbourhood theorem are obtained. Chapter 5 is devoted to several generalizations of semi- $C^*$ -Fredholm operators. In Chapter 6, closed range operators and their properties are considered. Semi-Fredholm operators on Hilbert spaces always have closed range, so the Hilbert  $C^*$ -module case requires more care. In particular,

necessary and sufficient conditions for a composition of two closed range operators to have closed image are given.

The remaining chapters deal with the generalization of the notion of the spectrum. In Chapter 7, a description of the generalized spectra of shift operators, unitary, selfadjoint, normal, Fredholm and semi-Fredholm operators on the standard Hilbert  $C^*$ -module are given. Chapter 8 contains results on perturbation of generalized spectra. Special attention is paid to the case when an operator is given by an upper triangular 2-by-2 matrix with operator entries. Chapter 9 is devoted to the study of relations between generalized spectra of an operator and of its compressions. Final Remarks remind the limitations of validity of the above results and emphasize the difference between the theory for Hilbert spaces and for (standard) Hilbert  $C^*$ -modules.

### 5. The conclusion and recommendation

S. Ivković's thesis develops the theory of semi-Fredholm operators on Hilbert  $C^*$ -modules. This gives a tool to study operators that are not (semi-)Fredholm in any reasonable sense, but become (semi-)Fredholm over an appropriate  $C^*$ -algebra. The results are very impressive, and will certainly find applications. The thesis is written clearly, contains all proofs and a lot of interesting examples. The author had to overcome a lot of technical difficulties in his work and showed himself as a specialist on Hilbert  $C^*$ -modules and on (semi-)Fredholm operators.

Due to all the above, we propose to the Teaching-Scientific Council of the Faculty of Mathematics to accept the attached text as a doctoral dissertation of Stefan Ivković and to appoint a commission for the defense.

Moscow,  
November 7,  
2021

### MEMBERS OF THE COMMITTEE

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