THE 4th ROMANIAN MASTER OF MATHEMATICS COMPETITION

DAY 2: SATURDAY, FEBRUARY 26, 2011, BUCHAREST

Language: English

Problem 4. Given a positive integer $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, we write $\Omega(n)$ for the total number $\sum_{i=1}^{s} \alpha_i$ of prime factors of n, counted with multiplicity. Let $\lambda(n) = (-1)^{\Omega(n)}$ (so, for example, $\lambda(12) = \lambda(2^2 \cdot 3^1) = (-1)^{2+1} = -1$).

Prove the following two claims:

i) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = +1$;
ii) There are infinitely many positive integers n such that $\lambda(n) = \lambda(n+1) = -1$.

Problem 5. For every $n \geq 3$, determine all the configurations of n distinct points $X_1, X_2, \ldots, X_n$ in the plane, with the property that for any pair of distinct points X_i, X_j there exists a permutation σ of the integers $\{1, \ldots, n\}$, such that $\mathrm{d}(X_i, X_k) = \mathrm{d}(X_j, X_{\sigma(k)})$ for all $1 \leq k \leq n$.
(We write $\mathrm{d}(X, Y)$ to denote the distance between points X and Y.)

Problem 6. The cells of a square 2011×2011 array are labelled with the integers $1, 2, \ldots, 2011^2$, in such a way that every label is used exactly once. We then identify the left-hand and right-hand edges, and then the top and bottom, in the normal way to form a torus (the surface of a doughnut).

Determine the largest positive integer M such that, no matter which labelling we choose, there exist two neighbouring cells with the difference of their labels at least M.*

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

*Cells with coordinates (x, y) and (x', y') are considered to be neighbours if $x = x'$ and $y - y' \equiv \pm 1 \pmod{2011}$, or if $y = y'$ and $x - x' \equiv \pm 1 \pmod{2011}$.