THE 4^{th} ROMANIAN MASTER OF MATHEMATICS COMPETITION

DAY 1: FRIDAY, FEBRUARY 25, 2011, BUCHAREST

Language: English

Problem 1. Prove that there exist two functions $f, g \colon \mathbb{R} \to \mathbb{R}$, such that $f \circ g$ is strictly decreasing and $g \circ f$ is strictly increasing.

Problem 2. Determine all positive integers n for which there exists a polynomial $f(x)$ with real coefficients, with the following properties:

(1) for each integer k, the number $f(k)$ is an integer if and only if k is not divisible by n;
(2) the degree of f is less than n.

Problem 3. A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.